$\mathcal{D}$–modules and Logarithmic Comparison Theorems

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Abstract.

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Let us denote X a complex manifold of dimension n, $\mathcal{O}_X$ the sheaf of holomorphic functions on X and $\mathcal{D}_X$ the sheaf of linear differential operators with holomorphic coefficients.

Roughly speaking, a $\mathcal{D}_X$–module $\mathcal{M}$ (with some good finiteness properties) represents a system of linear partial differential equations with holomorphic coefficients on X. We are able to apply geometric and algebraic methods in the study of such systems. This is the central objective of $\mathcal{D}$–module Theory with applications in Singularity Theory. $\mathcal{D}$–module Theory was initiated by M. Sato, M. Kashiwara, and the *Kyoto School*, in the 60's of the last century with main contributions of Z. Mebkhout, and A. Grothendieck, B. Malgrange, D.C. Spencer along with others.

The use of (co)homological methods to study the singularities of a complex hypersurface $D \subset X$ dates back to the pioneering work of H. Cartan, J. Leray, J.-P. Serre and A. Grothendieck, among others.

Grothendieck's Comparison Theorem states that the cohomology of the complex of meromorphic forms with poles on D (denoted $\Omega^\bullet_X(*D)$) is (isomorphic to) the singular cohomology of the complement $U := X \setminus D$, *i.e.* $H^\bullet(U;\mathbb{C})$.

The isomorphism between these two cohomology vector spaces is the *de Rham morphism* of "integration along cycles". Since each cohomology class in $H^\bullet(U;\mathbb{C})$ is represented by a meromorphic form, what can be said about the order of its pole along D?

The order of the pole question has been studied by several authors and, in particular, by Ph. Griffiths. P. Deligne proved that, if D is a normal crossing divisor, then each cohomology class on U corresponds, via the de Rham morphism, to a differential form in $\Omega^\bullet(\log D)$, the subcomplex of $\Omega^\bullet(*D)$ of *logarithmic meromorphic forms* with respect to D. Thus, for this class of divisors, the natural inclusion map

$$i(D) : \Omega^\bullet(\log D) \to \Omega^\bullet(*D)$$

is a quasi-isomorphism.

We say that a general divisor $D \subset X$ satisfies the *Logarithmic Comparison Theorem* if the inclusion map $i(D)$ is a quasi-isomorphism. For a general divisor D the notion of logarithmic meromorphic form was introduced by K. Saito. In the last 30 years, an important number of publications have addressed the Logarithmic Comparison Theorem. $\mathcal{D}$–module Theory plays a relevant role in these works, mainly because the complex $\Omega^\bullet(*D)$ is the de Rham complex of the $\mathcal{D}_X$–module $\mathcal{O}_X(*D)$ of meromorphic functions with poles on D. F.J. Calderón has proved that when the divisor D is *Koszul free*, the

complex $\Omega^\bullet(\log D)$ is a perverse sheaf, since it is the solution complex of the *holonomic* $\mathcal{D}_X$–module defined by the *logarithmic vector fields* with respect to D. F.J. Calderón Moreno and L. Narváez Macarro have proved several versions of the Logarithmic Comparison Theorem by using tools of $\mathcal{D}$-module Theory.

In this talk, I will review some of the previous results. This work is partially supported by PID2020-117843GB-I00, and PID2024-156912NB-I00 and FEDER.