Projective crossed modules in semi-abelian categories

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Abstract.

We characterize the projective objects in the category of internal crossed modules [4, 5] within any (Janelidze–Márki–Tholen [6]) semi-abelian category. When this category forms a variety of algebras, the internal crossed modules again constitute a semi-abelian variety, ensuring the existence of free objects, and thus of enough projectives. We show that such a variety is not necessarily Schreier—subobjects of free objects are again free—, but does satisfy the so-called Condition (P) [3]—meaning the class of projectives is closed under protosplit subobjects—if and only if the base variety satisfies this condition. As a consequence, the non-additive left chain-derived functors of π_0, the connected components functor, are well defined (in the sense of [3]) in this context.

This presentation aims to revisit key results regarding projective and free crossed modules over groups, as established in [1]. It will subsequently demonstrate how these results are employed to construct the non-additive left derived functors for $\pi_0 \colon \mathsf{XMod} \to \mathsf{Gp}$, following the framework outlined in [3]. Lastly, the discussion will be generalized to encompass $\pi_0 \colon \mathsf{XMod}(\mathcal{V}) \to \mathcal{V}$, where $\mathcal{V}$ is a semi-abelian variety of algebras satisfying Condition (P), as referenced in [2].

References

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