Experimental Mathematics: Using Symbolic Computation to Analyze Conjectures Involving Matrices and Polynomials

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Abstract.

The availability of Symbolic Computation tools enables us to introduce an experimental approach into the research we do in mathematics, when appropriate. I will present three concrete examples where the use of these tools has been essential to discover, conjecture and, in some cases, demonstrate new properties of the mathematical objects that we were considering - mostly polynomials and matrices.

P-matrices are matrices all of whose principal minors are positive. Q-matrices are matrices whose sums of principal minors of the same order are positive. A matrix is a PM-matrix if all its powers are P-matrices. A matrix is a QM-matrix if all its powers are Q-matrices. The study of the eigenvalues of these matrices brings many open questions. For example, until 2024 (see [6]), it was not known if the eigenvalues of a PM-matrix were necessarily positive (solving a longstanding conjecture raised in [5]). Or it is not known if the eigenvalues of a matrix A such that A and A^2 are P-matrices necessarily have positive real parts (see [4]). Taking into account that a P (resp. PM) matrix is a Q (resp. QM) matrix, we will study these questions for Q-matrices and QM-matrices in order to find an answer for the original problems. By using Symbolic Computation, we will show how to characterise the real QM-matrices up-to size 5 and we characterise those real matrices A, 4×4, such that A and A^2 are Q-matrices but not all eigenvalues of A have positive real part.

The second example will illustrate how Symbolic Computation helps to generate and to characterize correlation matrices when their entries come from a finite set. Bohemian matrices are families of matrices whose entries come from a fixed discrete set of small integers. A symmetric matrix is a correlation matrix if it has ones on the diagonal and its eigenvalues are nonnegative. We will introduce a new characterization of correlation matrices (based on Descartes Law of Signs, see [3]) that will be used to show that the number of Bohemian Correlation Matrices over -1, 0 and 1 corresponds to Bell or exponential numbers, the number of ways to partition a set of n labeled elements (see `https://oeis.org/A000110`), and to solve some correlation matrix completion problems.

Last example will be devoted to introducing the use of Symbolic Computation for trying to prove (or disprove) a conjecture about the spread of a symmetric matrix (i.e. the maximum absolute value of the difference between any two eigenvalues) with entries in the closed interval $[a, b]$. Let A be a square matrix with entries in $\mathbb{R}$. The spread of A is defined as the maximum of the distances between the eigenvalues of A. Let $S_m[a, b]$ denote the set of all $m \times m$ symmetric matrices with entries in the real interval $[a, b]$ and let $S_m\{a, b\}$ be the subset of $S_m[a, b]$ of Bohemian matrices with population from only the extremal elements $\{a, b\}$. S. M. Fallat and J. J. Xing in 2012 proposed the following conjecture: the maximum spread in $S_m[a, b]$ is attained by a rank 2 matrix in $S_m\{a, b\}$. X. Zhan had proved previously that the conjecture was true for $S_m[-a, a]$ with $a > 0$. We will show how

to interpret this problem geometrically, via polynomial resultants, in order to be able to treat this conjecture from a computational point of view. This will allow us to prove that this conjecture is true for several formerly open cases (see [1]).

References

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