Intrinsic tensor products and a Ganea-type extension of the five-term exact sequence

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Abstract.

In the context of group homology, Ganea extended the five-term exact homology sequence with an additional term involving a tensor product [12]. Work by Arias, Casas, Ladra, Pirashvili and others [17, 10, 5, 7, 19, 6, 1] investigates versions of this theorem in diverse settings, such as (pre)crossed modules, associative algebras, Lie and Leibniz algebras. The aim of this talk, based on the preprint [8], is to explain how and when these results can be understood as instances of a general categorical phenomenon, founded on a characterisation of the tensor product in terms of limits and colimits, relying on tools borrowed from abstract polynomial functor calculus [2, 15, 13].

We define an intrinsic symmetric bi-right-exact (and for varieties, bi-cocontinuous) bilinear product on objects of a semi-abelian category [16], constructed as the cosmash product [4, 14] in the two-nilpotent reflection. When applied to abelian objects, this recovers classical tensor products in many cases (the oldest known one being the case of groups [18]). A recognition theorem states that any symmetric bi-cocontinuous bifunctor on an abelian variety of algebras is realised as the bilinear product in the variety of algebras over a suitable two-nilpotent symmetric operad.

This rests on a right-exactness theorem for cross-effects of bifunctors, and consequently for cosmash products. We develop basic properties, compare the bilinear product to the Brown–Loday non-abelian tensor product [3, 9], and prove a categorical Ganea-type six-term exact homology sequence, extending the five-term exact sequence of [11]. We further obtain explicit descriptions of bilinear products in categories of representations; in particular, the bilinear product of the associated Beck modules generalises the classical tensor product of representations for groups and Lie algebras.

This is joint work with Bo Shan Deval and Manfred Hartl.

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