Cohomology of Presheaves of Monoids

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Categorical, homological and computational methods in Algebra
Santiago de Compostela, June 24-26, 2026

Motivation

In this talk I will show a cohomolgy theory for presheaves of commutative monoids over a small category C

$$\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \textbf{Mon}$$

which we use to give a precise classification theorem for prestacks or contravariant pseudo-functors on the small category C with values in the 2-category of monoidal abelian groupoids:

$$\mathfrak{P} : \mathrm{C}^{\mathrm{op}} \to \textit{Monoidal Abelian Groupoids}.$$

In other terms, by Grothendieck's equivalence, a classification theorem for monoidal C-fibred categories whose fibres are monoidal abelian groupoids.That is, categories $\mathfrak{G}$ endowed with

- A (Grothendieck) fibration $\pi : \mathfrak{G} \to \mathrm{C}$.
- A C-fibred monoidal structure on $\mathfrak{G}$ (by means of cartesian C-functors $\otimes : \mathfrak{G} \times_{\mathrm{C}} \mathfrak{G} \to \mathfrak{G}$ and $\iota : \mathrm{C} \to \mathfrak{G}$) and,
- such that every fibre category $\mathfrak{G}_U = \pi^{-1}(1_U)$, $U \in Ob\mathrm{C}$, is a monoidal abelian groupoid

Abelian monoidal groupoids, categorical groups

-Recall that an *abelian groupoid* is a groupoid $\mathbb{G}$ whose isotropy groups $\mathrm{Aut}_{\mathbb{G}}(x)$, $x \in \mathrm{Ob}\mathbb{G}$, are all abelian.

-A *monoidal abelian groupoid* $\mathbb{G} = (\mathbb{G}, \otimes, \iota, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$ consists of an abelian groupoid $\mathbb{G}$ enriched with a tensor product $\otimes : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$, a unit object ι, and natural isomorphisms

$$\boldsymbol{a}_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z), \quad \boldsymbol{l}_x : \iota \otimes x \to x, \quad \boldsymbol{r}_x : x \otimes \iota \to x,$$

satisfying the "pentagon coherence condition" and the "triangle coherence condition"

-A *categorical group* is a monoidal groupoid such that any object is invertible w.r.t the tensor product.

Sinh's classification theorem [Ph.D. Thesis, 1975]

Categorical groups are examples of monoidal abelian groupoids since, if $\mathbb{G}$ is a categorical group then

- $\mathrm{Aut}_{\mathbb{G}}(\iota) \cong \mathrm{Aut}_{\mathbb{G}}(x)$ by $a \mapsto \boldsymbol{r}_x(id_x \otimes a)\boldsymbol{r}_x^{-1}$.
- $\mathrm{Aut}_{\mathbb{G}}(\iota)$ is abelian since $\mathrm{Aut}_{\mathbb{G}}(\iota) \times \mathrm{Aut}_{\mathbb{G}}(\iota) \to \mathrm{Aut}_{\mathbb{G}}(\iota)$, $(a, b) \mapsto \boldsymbol{r}_\iota(a \otimes b)\boldsymbol{l}_\iota^{-1}$ is a group homomorphism (Eckmann-Hilton)

Shin's classification theorem establishes that a complete invariant for the equivalence class of a categorical group $\mathbb{G}$ (w.r.t. monoidal equivalences) is given by a triple of data (G, A, c), where

- $G = \Pi_0\mathbb{G}$ is the group of isomorphism classes of its objects, with multiplication induced by the tensor product,
- $A = \Pi_1\mathbb{G} = \mathrm{Aut}_{\mathbb{G}}(\iota)$ with action $\theta : G \to Aut(A)$ canonically defined from the unit constraints
- c is a cohomology class $c \in H^3_{E-M}(G, A)$, which is canonically defined from the associativity constraint.

Extension to monoidal abelian groupoids

For general monoidal abelian groupoids, their cohomological classification was given by

Calvo, M.; Cegarra, A.M.; Heredia, B.A. Structure and classification of monoidal groupoids. *Semigroup Forum* (2008).

where it is shown how monoidal abelian groupoids are classified by elements of the third Leech cohomology group of monoids $H^3_{Leech}(M, A)$. In that classification process, for each monoidal abelian groupoid $\mathbb{G}$, M is its monoid of connected components, with multiplication induced by the tensor product, the coefficients $A : D(M) \to \mathbf{Ab}$ is a functor from the category of factorization of the monoid M that is provided by its automorphism groups, and the classifying datum $c \in H^3_{Leech}(M, A)$ is the cohomology class of a certain 3-cocycle canonically constructed from its structure associativity constraint.

Relevant cases

$$\mathfrak{P} : \mathrm{C}^{\mathrm{op}} \to \mathit{Mon\ Abel\ Groupoids} \overset{\mathit{Conne.\ Comp.}}{\rightsquigarrow} \mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \mathbf{Mon}$$

- $\mathrm{C} = O(X)$, the category defined by the partially ordered set of open subsets of a topological space X. That is, in the cohomology of presheaves of monoids (and presheaves of groups, then) on a topological space.
- $\mathrm{C} = \Delta$, the simplicial category of finite ordered sets $[p] = \{0 < 1 < \cdots < p\}$, with non-decreasing maps between them as its morphisms. That is, in the cohomology of simplicial monoids.
- $\mathrm{C} = O(X) \times \Delta$, where X is a topological space. That is, in the cohomology of presheaves of simplicial monoids on a topological space (or, equivalently, of simplicial presheaves of monoids on a topological space).
- $\mathrm{C} = Or(G) \times \Delta$, where $Or(G)$ is the orbit category of a group G, whose objects are the transitive left G-sets G/H, for any subgroup $H \subseteq G$, and whose morphisms are the G-equivariant maps between them. That is, in the equivariant cohomology of simplicial monoids endowed with a left G-action by automorphisms. Here, one regards such a simplicial monoid $\mathcal{M}$ as the presheaf of monoids on $Or(G) \times \Delta$ such that $(G/H, [p]) \mapsto \mathrm{Hom}_G(G/H, \mathcal{M}_p) \cong \mathcal{M}_p^H$.

Cohomology of small categories and Gabriel-Zisman cohomology of simplicial sets.

♦ Let K be a small category,

The category of K-modules $\mathrm{K}\text{-Mod} := \mathit{Func}(K, \mathbf{Ab})$.

K-Mod is an abelian category with enough projective and injective objects.

For $A : \mathrm{K} \to \mathbf{Ab}$ a K-module, the *cohomology groups* of K with coefficients in a A, are defined as

$$H^n(\mathrm{K}, A) := \mathrm{Ext}^n_{\mathrm{K}}(\mathbb{Z}, A),$$

where $\mathbb{Z} : \mathrm{K} \to \mathbf{Ab}$ is the constant functor defined by the abelian group of integers $\mathbb{Z}$.

♦ For $X : \Delta^{op} \to \mathit{Set}$ a simplicia set, its *category of simplices*

$$\Delta(X) \ : \begin{cases} Ob\Delta(X) & \text{are} \quad \text{simplices } u \in X \\ \alpha : u \to v & \text{are} \quad \alpha : [n] \to [m] \in \Delta \text{ with } u = \alpha^* v. \end{cases}$$

A $\Delta(X)$-module, $A : \Delta(X) \to \mathbf{Ab}$, is called a *coefficient system on a X*.

The Gabriel and Zisman cohomology groups of X with coefficients in A , are defined as

$$H^n_{G-Z}(X, A) = H^n(\Delta(X), A) = \mathrm{Ext}^n_{\Delta(X)}(\mathbb{Z}, A).$$

Cohomology of small categories and Gabriel-Zisman cohomology of simplicial sets II

A standar cochain complex $C^\bullet(X,A)$, for the calculation of the Grabiel-Zisman cohomology, consists of the abelian groups

$$C^p(X,A) = \prod_{x \in X_p} A(x)$$

with $\partial : C^{p-1}(X,A) \to C^p(X,A)$ given by

$$(\partial\varphi)(x) = \sum_{i=0}^{p} (-1)^i A(d^i)(\varphi(d_i x)).$$

Then, By Gabriel–Zisman [Calculus of fractions and homotopy theory, (1967). Appendix II, Prop. 3.3.], there are natural isomorphisms

$$H^n_{G-Z}(X,A) \cong H^n C^\bullet(X,A) \qquad (n \geq 0).$$

Cohomology of small categories and Gabriel-Zisman cohomology of simplicial sets III

Recall that the nerve $\mathbf{N}\mathrm{K}$ of a small category K is the simplicial set whose p-simplices are sequences

$$\sigma = (\sigma 0 \stackrel{\sigma_1}{\to} \cdots \stackrel{\sigma_p}{\to} \sigma p)$$

of p composable morphisms in K (objects $\sigma 0$ of K if $p = 0$), with face and degeneracy operators,

$$\begin{aligned} d_0\sigma &= (\sigma_2, \ldots, \sigma_p)\ (\sigma 1 \,\text{if}\, p = 1) \\ d_i\sigma &= (\sigma_1, \ldots, \sigma_{i+1}\sigma_i, \ldots, \sigma_p) \text{ if } 0 < i < p, \\ d_p\sigma &= (\sigma_1, \ldots, \sigma_{p-1})\ (\sigma 0 \,\text{if}\, p = 1). \\ s_j\sigma &= (\sigma_1, \ldots, \sigma_j, 1_{\sigma j}, \sigma_{j+1}, \ldots, \sigma_p). \end{aligned}$$

There is a canonical last object functor $L : \Delta(\mathbf{N}\mathrm{K}) \to \mathrm{K},\ \sigma \mapsto \sigma p$, and then, by composing with it, every K-module $A : K \to \mathbf{Ab}$ defines a system of coefficients on $\mathbf{N}\mathrm{K}$, $AL : \Delta(NK) \to \mathbf{Ab}$. We prove

$$H^n(\mathrm{K}, A) \cong H^n_{G-Z}(\mathbf{N}\mathrm{K}, AL).$$

Coefficients for the cohomology of presheaves monoids

Let C be a fixed small category

$$\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \textbf{Mon} : \begin{cases} U \in Ob\mathrm{C} & \mapsto & \mathcal{M}(U), \\ \sigma : V \to U & \mapsto & (\ )^{\sigma} : \mathcal{M}(U) \to \mathcal{M}(V). \end{cases}$$

If M is a monoid, its Leech *category of factorizations*, denoted by $\mathrm{D}(M)$,

$$\mathrm{D}(M) \ : \begin{cases} Ob\mathrm{D}(M) & \text{are} & \text{elements } x \in M \\ (u_0, u_1) : x \to y & \text{are} & (u_0, u_1) \in M^2 \text{ with } u_0 x u_1 = y. \end{cases}$$

This construction defines a functor $\mathrm{D} : \textbf{Mon} \to \textbf{Cat}$. By composing with D, every presheaf $\mathcal{M}$ defines a diagram of categories $\mathrm{D}\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \textbf{Cat}$. Let

$$\mathrm{D}(\mathcal{M}) = \int_{\mathrm{C}} \mathrm{D}\mathcal{M} : \begin{cases} (U, x), \ U \in Ob\mathrm{C} \text{ and } x \in \mathcal{M}(U) \\ (\sigma, v_0, v_1) : (U, x) \to (V, y) \text{ where} \\ \sigma : V \to U \in \mathrm{C}, \ \text{and } v_0, v_1 \in \mathcal{M}(V) \text{ such that } v_0\, x^{\sigma}\, v_1 = y. \end{cases}$$

Then we prove that there is a natural equivalence

$$\textbf{Ab}(Psh(\mathrm{C}, \textbf{Mon})\!\downarrow_{\mathcal{M}}) \simeq \mathrm{D}(\mathcal{M})\text{-Mod}$$

therefore $\mathrm{D}(\mathcal{M})$-modules naturally arise as coefficients for the cohomology of a presheaf of a monoid $\mathcal{M}$.

Derivations of presheaves of monoids

We actually define two cohomology theories. One of them is given by the right derived functors os the derivation functor we next define:

Definition
A *derivation* of a presheaf $\mathcal{M} : \mathrm{C}^{op} \to \mathbf{Mon}$ in a $\mathrm{D}(\mathcal{M})$-module $A : \mathrm{D}(\mathcal{M}) \to \mathbf{Ab}$, denoted by $d : \mathcal{M} \to A$, is a function that assigns to each pair (U, x), where $U \in \mathrm{Ob}\mathrm{C}$ and $x \in \mathcal{M}(U)$, an element $d(U, x) \in A(U, x)$ satisfying

(*i*) $d(U, xy) = x\, d(U, y) + d(U, x)\, y$, for any $U \in \mathrm{Ob}\mathrm{C}$ and $x, y \in \mathcal{M}(U)$,

(*ii*) $d(U, x)^\sigma = d(V, x^\sigma)$, for any arrow $\sigma : V \to U$ in C and $x \in \mathcal{M}(U)$,

where, for $b \in A(U, y)$, $xb = A((1_u, x, e)(b)$ and for $a \in A(U, x)$, $ax = A(1_U, e, y)(a)$ and $a^\sigma = A((\sigma, e, e)(a)$ and e denotes the identity of the monoid $\mathcal{M}(U)$.

Under pointwise addition, the set of all derivations may be given an abelian group structure and, in fact, we have a functor

$$\mathrm{Der}(\mathcal{M}, -) : \mathrm{D}(\mathcal{M})\text{-Mod} \to \mathbf{Ab},$$

which we prove is representable (and then left exact), by the (suitably defined) *augmentation ideal*, that is $\mathrm{Der}(\mathcal{M}, -) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{M})}(I\mathcal{M}, -)$

Cohomologies for presheaves of monoids

Definition
For $\mathcal{M}$ a presheaf of monoids on C and $n \geq 0$, we define the *n-th cohomology group* of $\mathcal{M}$ with coefficients in a $\mathrm{D}(\mathcal{M})$-module A as the

$$H^n(\mathcal{M}, A) = H^n(\mathrm{D}(\mathcal{M}), A) = \mathrm{Ext}^n_{\mathrm{D}(\mathcal{M})}(\mathbb{Z}, A).$$

And, for each integer $n \geq 1$, we define the *n-th simple cohomology group* of $\mathcal{M}$ with coefficients in a A by

$$H^n_s(\mathcal{M}, A) = R^{n-1}\mathrm{Der}(\mathcal{M}, -)(A) = \mathrm{Ext}^{n-1}_{\mathrm{D}(\mathcal{M})}(I\mathcal{M}, A).$$

The terminology "simple" is taken from Gerstenhaber-Schack [Deformations of Algebra Morphisms and diagrams. Transc. A.M.S. (1983)]

We prove that for any $\mathrm{D}(\mathcal{M})$-module A there is a natural long exact sequence

$$\cdots \to H^n(\mathcal{M}, A) \to H^n(\mathrm{C}^{\mathrm{op}}, A) \to H^{n+1}_s(\mathcal{M}, A) \to H^{n+1}(\mathcal{M}, A) \to H^{n+1}(\mathrm{C}^{\mathrm{op}}, A) \to \cdots$$

linking the cohomology groups of $\mathcal{M}$ in both theories with those of the category C^{op} with coefficients in the C^{op}-module obtained by restriction of A to C^{op} through

$$\mathrm{C}^{op} \hookrightarrow \mathrm{D}(\mathcal{M}) : \ \sigma : V \to U \ \mapsto \ (\sigma, e, e) : (U, e) \to (V, e).$$

Cohomologies for presheaves of monoids II

When C^{op} is cohomology trivial, for instance whenever C has a final object , we deduce the existence of natural isomorphisms

$$H_s^n(\mathcal{M}, A) \cong H^n(\mathcal{M}, A), \qquad (n \geq 2).$$

Example

- When C is the final category, then $\mathcal{M} = M$ is simply a monoid, and

  $$H^n(M.-) \cong H^n_{Leech}(M, -)$$

  and $H_s^n(M.-) \cong H^n(M, -)$ for all $n \geq 2$.
- When $\mathrm{C} = G$, is a group and $\mathcal{M} = H$ is a right G-group, then $\mathrm{D}(\mathcal{M}) = H \rtimes G$, the semidirect product group, and

  $$H^n(\mathcal{M}, -) = H^n_{E-M}(H \rtimes G, -)$$

  whereas

  $$H_s^n(\mathcal{M}, -) = H^n_{n-1}(G, H; -),$$

  the vector cohomology introduced by Whitehead [On group extensions with operators. Quart. J. Math. (1950)].

A cochain complex for computations

We show a workable cochain complex to compute the cohomology groups, which comes from the homotopy colimit construction by Bousfield–Kan [Homotopy Limits, Completions and Localizations, Springer (1972)]

First we regard each monoid Γ as a small category with only one object. Then, the simplicial set $\mathsf{N}\Gamma$ is just its classifying space, that is, the reduced simplicial set whose p-simplices $\tau = (* \overset{\tau_1}{\to} \cdots \overset{\tau_p}{\to} *) = (\tau_1, \dots, \tau_p)$ are the elements of Γ^p.

From $\mathcal{M} : \mathrm{C}^{op} \to \mathbf{Mon} \ \rightsquigarrow \ \mathsf{N}\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \mathbf{SSet}$.

Let $\Psi(\mathcal{M})$ be the simplicial replacement construction by Bousfield-Kan on $\mathsf{N}\mathcal{M}$; that is, the bisimplicial set whose (p, q)-bisimplices

$$\Psi_{p,q}(\mathcal{M}) = \coprod_{\beta = (\beta 0 \overset{\beta_1}{\leftarrow} \cdots \overset{\beta_p}{\leftarrow} \beta p) \in \mathsf{N}_p \mathrm{C}^{\mathrm{op}}} \mathsf{N}_q \mathcal{M}(\beta 0) (= \mathcal{M}(\beta 0)^q).$$

The vertical face and degeneracy operators are those of the simplicial sets $\mathsf{N}\mathcal{M}(\beta 0)$, and the horizontal ones those of $\mathsf{N}\mathrm{C}^{\mathrm{op}}$, except $d_0^h : \Psi_{p,q}(\mathcal{M}) \to \Psi_{p-1,q}(\mathcal{M})$ which is defined by $d_0^h(\beta; \tau) = (d_0 \beta, \tau^{\beta_1})$.

A cochain complex for computations II

Let $\Delta^2\Psi(\mathcal{M})$ denote the category of bisimplices. There is a last object functor

$$\Delta^2\Psi(\mathcal{M}) \to \mathrm{D}(\mathcal{M}),\ (\beta,\tau) \mapsto (\beta p, (\tau_1\cdots\tau_q)^{\beta_1\cdots\beta_p})$$

and, then, by composition with it, every $\mathrm{D}(\mathcal{M})$-module A defines a system of coefficients on $\Psi(\mathcal{M})$ and gives rise to a bicosimplicial abelian group

$$C^{\bullet,\bullet}(\mathcal{M},A),\text{ where } C^{p,q}(\mathcal{M},A) = \prod_{(\beta,\tau)\in\Psi_{p,q}(\mathcal{M})} A\big(\beta p, (\tau_1\cdots\tau_q)^{\beta_1\cdots\beta_p}\big)$$

Let also write $C^{\bullet,\bullet}(\mathcal{M},A)$ for its alternating faces sum cochain bicomplex, whose horizontal and vertical coboundaries are

$$\partial_h = \sum_{i=0}^{p}(-1)^i d_h^i : C^{p-1,q}(\mathcal{M},A) \to C^{p,q}(\mathcal{M},A),$$

$$\partial_v = (-1)^p\sum_{j=0}^{q}(-1)^j d_v^j : C^{p,q-1}(\mathcal{M},A) \to C^{p,q}(\mathcal{M},A).$$

We define the *complex of cochains of $\mathcal{M}$ with coefficients in a $\mathrm{D}(\mathcal{M})$-module A* as $C^{\bullet}(\mathcal{M},A) = \mathrm{Tot}\, C^{\bullet,\bullet}(\mathcal{M},A)$,

$$C^n(\mathcal{M},A) = \bigoplus_{p+q=n} C^{p,q}(\mathcal{M},A), \qquad \partial = \partial_h + \partial_v : C^{n-1}(\mathcal{M},A) \to C^n(\mathcal{M},A).$$

Theorem

Let $\mathcal{M}$ be a presheaf of monoids on C. For any $\mathrm{D}(\mathcal{M})$-module A there are natural isomorphisms

$$H^n(\mathcal{M}, A) \cong H^n C^\bullet(\mathcal{M}, A), \qquad n \geq 0.$$

For the calculation of $H_s^n(\mathcal{M}, A)$ we consider a subcomplex of

$$C_s^\bullet(\mathcal{M}, A) \subset C^\bullet(\mathcal{M}, A)$$

where $C_s^0(\mathcal{M}, A) = 0$ and for $n \geq 1$

$$C_s^n(\mathcal{M}, A) = \{\varphi \in C^n(\mathcal{M}, A) \mid \varphi|_{\Psi_{n,\mathbf{0}}} = 0\} = \bigoplus_{\substack{p+q=n \\ q \geq 1}} C^{p,q}(\mathcal{M}, A)$$

Theorem

$$H_s^n(\mathcal{M}, A) \cong H^n C_s^\bullet(\mathcal{M}, A), \qquad n \geq 1.$$

A cochain complex for computations IV

Notice that the homotopy colimit of $\mathbf{N}\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to$ **SSet** is the simplicial set diagonal of $\Psi(\mathcal{M})$:

$$\operatorname{hocolim} \mathbf{N}\mathcal{M} = \operatorname{diag}\Psi(\mathcal{M}).$$

Then, every $\mathrm{D}(\mathcal{M})$-module A defines a coefficient system on $\mathrm{hocolim}\mathbf{N}\mathcal{M}$ and

$$H^n_{G-Z}(\operatorname{hocolim} \mathbf{N}\mathcal{M}, A) = H^n C^\bullet(\operatorname{hocolim} \mathbf{N}\mathcal{M}, A),$$

where $C^\bullet(\operatorname{hocolim} \mathbf{N}\mathcal{M}, A)$ is the alternating faces sum cochain complex of $\operatorname{diag} C^{\bullet,\bullet}(\mathcal{M}, A)$, The generalized Eilenberg-Zilber theorem of Dold-Puppe (Theorem 2.4 in Goerss-Jardine [Simplicial Homotopy Theory Birkhauser (2009)]) shows that both cochain complexes $C^\bullet(\mathcal{M}, A)$ and $C^\bullet(\mathrm{hocolim}\mathbf{N}\mathcal{M}, A)$ are cohomology equivalent in a natural way.

Theorem

Let $\mathcal{M}$ be a presheaf of monoids on C. For any $\mathrm{D}(\mathcal{M})$-module A, there are natural isomorphisms

$$H^n(\mathcal{M}, A) \cong H^n_{G-Z}(\operatorname{hocolim} \mathbf{N}\mathcal{M}, A).$$

and if C^{op} is cohomology trivial

$$H^n_s(\mathcal{M}, A) \cong H^n(\mathcal{M}, A) \cong H^n_{G-Z}(\operatorname{hocolim} \mathbf{N}\mathcal{M}, A), \qquad (n \geq 2).$$

Low dimensional simple cochains

The second and third simple cohomology groups give our classification results and then. the relevant truncated subcomplex of the complex $C_s^\bullet(\mathcal{M}, A)$ is

$$0 \to C_s^1(\mathcal{M}, A) \overset{\partial}{\to} C_s^2(\mathcal{M}, A) \overset{\partial}{\to} C_s^3(\mathcal{M}, A) \overset{\partial}{\to} C_s^4(\mathcal{M}, A)$$

where:

♦ A 1-cochain $f \in C_s^1(\mathcal{M}, A)$ is a function assigning an element

$$f(U; x) \in A(U, x), \quad \text{to each object } U \text{ of } \mathrm{C} \text{ and } x \in \mathcal{M}(U).$$

♦ A 2-cochain $g \in C_s^2(\mathcal{M}, A)$ is a function assigning elements

$$\begin{cases} g(U; x, y) \in A(U, xy), & \text{to each object } U \text{ of } \mathrm{C} \text{ and } x, y \in \mathcal{M}(U), \\ g(\alpha; x) \in A(U_1, x^\alpha), & \text{to each arrow } U_0 \overset{\alpha}{\leftarrow} U_1 \text{ of } \mathrm{C} \text{ and } x \in \mathcal{M}(U_0). \end{cases}$$

♦ A 3-cochain $h \in C_s^3(\mathcal{M}, A)$ is a function assigning elements

$$\begin{cases} h(U; x, y, z) \in A(U, xyz), & \text{to each object } U \text{ of } \mathrm{C} \text{ and } x, y, z \in \mathcal{M}(U), \\ h(\alpha; x, y) \in A(U_1, x^\alpha y^\alpha), & \text{to each arrow } U_0 \overset{\alpha}{\leftarrow} U_1 \text{ of } \mathrm{C} \text{ and } x, y \in \mathcal{M}(U_0), \\ h(\alpha, \beta; x) \in A(U_2, x^{\alpha\beta}), & \text{to each arrows } U_0 \overset{\alpha}{\leftarrow} U_1 \overset{\beta}{\leftarrow} U_2 \text{ of } \mathrm{C} \text{ and } x \in \mathcal{M}(U_0). \end{cases}$$

Low dimensional simple cochains II

- The coboundary $\partial : C^1_s(\mathcal{M}, A) \to C^2_s(\mathcal{M}, A)$ acts on an 1-cochain f by

$$(\partial f)(U; x, y) = x\, f(U; y) - f(U; xy) + f(U; x)y,$$
$$(\partial f)(\alpha; x) = f(U_1; x^\alpha) - f(U_0; x)^\alpha.$$

- The coboundary $\partial : C^2_s(\mathcal{M}, A) \to C^3_s(\mathcal{M}, A)$ acts on a 2-cochain g by

$$(\partial g)(U; x, y, z) = x\, g(U; y, z) - g(U; xy, z) + g(U; x, yz) - g(U; x, y)\, z,$$
$$(\partial g)(\alpha; x, y) = g(U_1; x^\alpha, y^\alpha) - g(U_0; x, y)^\alpha - x^\alpha g(\alpha; y) + g(\alpha; xy) - g(\alpha; x)\, y^\alpha$$
$$(\partial g)(\alpha, \beta; x) = g(\beta; x^\alpha) - g(\alpha\beta; x) + g(\alpha; x)^\beta.$$

- The coboundary $\partial : C^3_s(\mathcal{M}, A) \to C^4_s(\mathcal{M}, A)$ acts on a 3-cochain h by

$$(\partial h)(U; x, y, z, t) = x\, h(U; y, z, t) - h(U; xy, z, t) + h(U; x, yz, t) - h(U; x, y, zt) + h(U; x, y, z)\, t,$$

$$(\partial h)(\alpha; x, y, z) = h(U_1; x^\alpha, y^\alpha, z^\alpha) - h(U_0; x, y, z)^\alpha - x^\alpha h(\alpha; y, z) + h(\alpha; xy, z) - h(\alpha; x, yz) + h(\alpha; x, y)z^\alpha,$$

$$(\partial h)(\alpha, \beta; x, y) = h(\beta; x^\alpha, y^\alpha) - h(\alpha\beta; x, y) + h(\alpha; x, y)^\beta + x^{\alpha\beta} h(\alpha, \beta; y) - h(\alpha, \beta; xy) + h(\alpha, \beta; x)\, y^{\alpha\beta},$$

$$(\partial h)(\alpha, \beta, \gamma; x) = h(\beta, \gamma; x^\alpha) - h(\alpha\beta, \gamma; x) + h(\alpha, \beta\gamma; x) - h(\alpha, \beta; x)^\gamma.$$

An *extension* of $\mathcal{M} : \mathrm{C}^{op} \to \textbf{Mon}$ is a *locally surjective* morphism of presheaves of monoids $\mathfrak{f} : \mathcal{E} \to \mathcal{M}$ (that is, $\mathfrak{f}_U : \mathcal{E}(U) \to \mathcal{M}(U)$ is surjective $\forall U \in Ob\mathrm{C}$).

If $A : \mathrm{D}(\mathcal{M}) \to \textbf{Ab}$ is a $\mathrm{D}(\mathcal{M})$-module, an *extension* $\bar{\mathcal{E}} = (\mathcal{E}, \mathfrak{f}, +)$ *of* $\mathcal{M}$ *by* A is an extension $\mathfrak{f} : \mathcal{E} \to \mathcal{M}$ of $\mathcal{M}$ endowed, for each $U \in \mathrm{ObC}$ and $x \in \mathcal{M}(U)$, with a simply-transitive action

$$+ : A(U, x) \times \mathfrak{f}_U^{-1}(x) \to \mathfrak{f}_U^{-1}(x), \qquad (a, w) \mapsto a + w,$$

such that

$$(a + w)(a' + w') = a\,x' + x\,a' + ww', \qquad (a + w)^\sigma = a^\sigma + w^\sigma,$$

for any object U of C, $x, x' \in \mathcal{M}(U)$, $\omega \in \mathfrak{f}_U^{-1}(x)$, $\omega' \in \mathfrak{f}_U^{-1}(x')$, $a \in A(U, x)$, $a' \in A(U, x')$, and any arrow $\sigma : V \to U$ of C.

Two such extensions of $\mathcal{M}$ by A, say $\bar{\mathcal{E}}$ and $\bar{\mathcal{E}}'$, are *isomorphic* if there exists an isomorphism of presheaves of monoids $\mathfrak{g} : \mathcal{E} \cong \mathcal{E}'$ such that $\mathfrak{f}'\mathfrak{g} = \mathfrak{f}$ and $\mathfrak{g}_U(a + w) = a + \mathfrak{g}_U(w)$, for any $U \in \mathrm{ObC}$, $w \in \mathcal{E}(U)$ and $a \in A(U, \mathfrak{f}_U(w))$.
Let

$$\mathrm{Ext}(\mathcal{M}, A)$$

denote the set of isomorphism classes $[\bar{\mathcal{E}}]$ of extensions $\bar{\mathcal{E}}$ of $\mathcal{M}$ by A.

For a given extension $\bar{\mathcal{E}} = (\mathcal{E}, \mathfrak{f} : \mathcal{E} \to \mathcal{M}, +)$ we can choose a family of section maps $S = (S_U : \mathcal{M}(U) \to \mathcal{E}(U))$, one for each $U \in \mathrm{Ob}\mathrm{C}$, such that $\mathfrak{f}_U S_U = id_{\mathcal{M}(U)}$. Then, a 2-cocycle $g = g_{\bar{\mathcal{E}},S} \in Z^2(\mathcal{M}, A)$ is defined as follows:

- for each object U of C and $x, y \in \mathcal{M}(U)$, let $g(U; x, y)$ be the element of $A(U, xy)$ determined by the equation

$$S_U(xy) = g(U; x, y) + S_U(x)\, S_U(y).$$

- for each arrow $U_0 \overset{\alpha}{\leftarrow} U_1$ of C and $x \in \mathcal{M}(U_0)$, let $g(\alpha; x)$ be the element of $A(U_1, x^\alpha)$ determined by the equation

$$(S_{U_0}(x))^\alpha = g(\alpha; x) + S_{U_1}(x^\alpha).$$

Then we obtain a simple 2-cocycle and the correspondence

$$\mathrm{Ext}(\mathcal{M}, A) \to H^2_s(\mathcal{M}, A)$$
$$[\bar{\mathcal{E}}] \mapsto [g_{\bar{\mathcal{E}},S}]$$

is a well defined map which gives a natural bijection and the classification of extensions.

Our results on the classification of prestacks over a small category C, by means of the third simple cohomology groups can be summarized as follows:

- If $\mathcal{M}$ is a presheaf of monoids on C and A is $\mathrm{D}(\mathcal{M})$-module, every simple 3-cocycle $h \in Z^3_s(\mathcal{M}, A)$ gives rise to a prestack $\mathfrak{P}(\mathcal{M}, A, h)$.
- For any prestack $\mathfrak{P}$ on C, there exist a presheaf of monoids $\mathcal{M}$, a $\mathrm{D}(\mathcal{M})$-module A, a simple 3-cocycle $h \in Z^3_s(\mathcal{M}, A)$, and an equivalence $\mathfrak{P}(\mathcal{M}, A, h) \simeq \mathfrak{P}$.
- If $h \in Z^3_s(\mathcal{M}, A)$ and $h' \in Z^3_s(\mathcal{M}', A')$ are simple 3-cocycles, then there is an equivalence $\mathfrak{P}(\mathcal{M}, A, h) \simeq \mathfrak{P}(\mathcal{M}', A', h')$ if and only if there is an isomorphism of presheaves of monoids $\mathfrak{f} : \mathcal{M}' \cong \mathcal{M}$ and an isomorphism of $\mathrm{D}(\mathcal{M}')$-modules $F : A' \cong \mathfrak{f}^* A$ such that the equality of cohomology classes $[h'] = F_*^{-1}\mathfrak{f}^*([h])$ in $H^3_s(\mathcal{M}', A')$ holds.

Thus, prestacks on C are classified by triples $(\mathcal{M}, A, c)$ where $\mathcal{M}$ is a presheaf of monoids on C, A is a $\mathrm{D}(\mathcal{M})$-module, and $c \in H^3_s(\mathcal{M}, A)$.

Classification of prestacks of monoidal abelian groupoids II

A prestack $\mathfrak{P}$, that is a contravariant pseudo-functors

$$\mathfrak{P} : \mathrm{C}^{\mathrm{op}} \to \textit{Monoidal Abelian Groupoids},$$

consists of the data (PDi) that follow.

(PD1) a monoidal abelian groupoid $\mathfrak{P}(U) = (\mathfrak{P}(U), \otimes, \iota, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$, for each object U of C

(PD2) a monoidal functor $\mathfrak{P}(\alpha) = (\ )^{\alpha} = ((\ )^{\alpha}, \phi^{\alpha}, \phi^{\alpha}_{\star}) : \mathfrak{P}(U_0) \to \mathfrak{P}(U_1)$, for each arrow $U_0 \xleftarrow{\alpha} U_1$ of C

(PD3) a monoidal transformation $\boldsymbol{\theta}^{\alpha,\beta} : ((\ )^{\alpha})^{\beta} \Rightarrow (\ )^{\alpha\beta}$, for each two arrows $U_0 \xleftarrow{\alpha} U_1 \xleftarrow{\beta} U_2$ of C

(PD4) a monoidal transformation $\boldsymbol{\theta}^{U} : id_{\mathfrak{P}(U)} \Rightarrow (\ )^{1_U}$, for each object U of C

These data are subject to the following coherence conditions:

$$\boldsymbol{\theta}^{\alpha,\beta\gamma}_{x} + \boldsymbol{\theta}^{\beta,\gamma}_{x^{\alpha}} = \boldsymbol{\theta}^{\alpha\beta,\gamma}_{x} + (\boldsymbol{\theta}^{\alpha,\beta}_{x})^{\gamma}. \quad \boldsymbol{\theta}^{\mathbf{1}_{U_0},\alpha}_{x} = -(\boldsymbol{\theta}^{U_0}_{x})^{\alpha}, \quad \boldsymbol{\theta}^{\alpha,\mathbf{1}_{U_1}}_{x} = -\boldsymbol{\theta}^{U_1}_{x^{\alpha}},$$

for any three composable arrows $U_0 \xleftarrow{\alpha} U_1 \xleftarrow{\beta} U_2 \xleftarrow{\gamma} U_3$ of C and $x \in \mathrm{Ob}\mathfrak{P}(U_0)$

It is not difficult to see that any prestack is equivalent to a skeletal prestack. That is one in which the groupoid $\mathfrak{P}(U)$ is skeletal for any object $U \in \mathrm{C}$. Then the associated triple $(\mathcal{M}, A, c)$ is described as follows

Classification of prestacks of monoidal abelian groupoids III

♦ *The presheaf of monoids* $\mathcal{M}:\mathrm{C}^{op}\to\mathbf{Mon}$:

- $U\in Ob\mathrm{C}\mapsto\mathcal{M}(U)=\mathrm{Ob}\mathfrak{P}(U)$ with $xy=x\otimes y$ (the existence of $\boldsymbol{a}_{x,y,z}$, $\boldsymbol{l}_x$, and $\boldsymbol{r}_x$ imply that $\mathcal{M}(U)$ is a monoid with $e=\iota$).

- $\alpha:V\to U\in Arr\mathrm{C}\mapsto\mathcal{M}(\alpha):\mathcal{M}(U)\to\mathcal{M}(V)$, the function on objects of $(\ )^{\alpha}:\mathfrak{P}(U)\to\mathfrak{P}(V)$ in (PD2) (the existence of $\phi^{\alpha}_{x,y}$ and $\phi^{\alpha}_{\star}$ imply that $\mathcal{M}(\alpha)$ in a homomorphism of monoids).

- Functoriality of $\mathcal{M}$ follows from the presence of the monoidal transformations $\boldsymbol{\theta}^{\alpha,\beta}_x$ in (PD3) and $\boldsymbol{\theta}^{U}_x:x\to x^{1_U}$ as data in (PD4).

♦ *The* $\mathrm{D}(\mathcal{M})$*-module* $A:\mathrm{D}(\mathcal{M})\to\mathbf{Ab}$:

- $(U\in Ob\mathrm{C},x\in\mathcal{M}(U))\mapsto A(U,x)=\mathrm{Aut}_{\mathfrak{P}(U)}(x)$

- $(\sigma,v_0,v_1):(U,x)\to(V,v_ox^{\sigma}v_1)\mapsto A(\sigma,v_0,v_1):A(U,x)\to A(V,v_ox^{\sigma}v_1)$ given by $a\mapsto(id_{\boldsymbol{v_0}}\otimes a^{\sigma})\otimes id_{\boldsymbol{v_1}}$

- The functoriality of A follows from the naturality of the associativity and unit constraints, the naturality of the structure morphisms in (PD2) and (PD3) and the abelianity of the groups $A(U,x)$.

♦ *The 3-cocycle* $h\in Z^3_s(\mathcal{M},A)$: This is defined by

$$\begin{cases} h(U;x,y,z)=\boldsymbol{a}_{x,y,z}, & \text{for each object } U \text{ of } \mathrm{C} \text{ and } x,y,z\in\mathcal{M}(U),\\ h(\alpha;x,y)=-\phi^{\alpha}_{x,y}, & \text{for each arrow } U_0\overset{\alpha}{\leftarrow}U_1 \text{ of } \mathrm{C} \text{ and } x,y\in\mathcal{M}(U_0),\\ h(\alpha,\beta;x)=\boldsymbol{\theta}^{\alpha,\beta}_x, & \text{each each arrows } U_0\overset{\alpha}{\leftarrow}U_1\overset{\beta}{\leftarrow}U_2 \text{ of } \mathrm{C} \text{ and } x\in\mathcal{M}(U_0).\end{cases}$$

- The cocycle condition $\partial h=0$ follow from the coherence conditions for the constrains involved in the data defining $\mathfrak{P}$.

When the monoids are groups

$$\mathrm{G} : \mathrm{C}^{op} \to Gp.$$

In this case, there is a full and faithful embedding

$$\mathrm{G}\text{-Mod} \hookrightarrow \mathrm{D}(\mathrm{G})\text{-Mod},$$

where a G-*module* is a presheaf of abelian groups on C ($= \mathrm{C}^{op}$-module) $A : \mathrm{C}^{op} \to \mathbf{Ab}$ such that

- The abelian group $A(U)$ is a left $\mathrm{G}(U)$-module, for each $U \in Ob\mathrm{C}$, and
- For each arrow $\sigma : V \to U$ of C the induced homomorphism $A(\sigma) = (\ )^{\sigma} : A(U) \to A(V)$ is a homomorphism of $\mathrm{G}(U)$-modules, where $A(V)$ is viewed as a $\mathrm{G}(U)$-module via the group homomorphism $G(\sigma) = (\ )^{\sigma} : \mathrm{G}(U) \to \mathrm{G}(V)$.

This embedding is an equivalence of categories. It identifies each G-module A to the $\mathrm{D}(\mathrm{G})$-module, equally denoted by A, such that $A(U, x) = A(U)$ for each object U of C and any $x \in \mathrm{G}(U)$, and $v_0\, a^{\sigma}\, v_1 = v_0 \cdot a^{\sigma}$, for each $\sigma : V \to U$ in C and $v_0, v_1 \in \mathrm{G}(V)$.

An extension of a presheaf of groups G by a G-module $A : \mathrm{C}^{op} \to \mathbf{Ab}$ is the same thing as a *singular extension of* G *by a* A defined as a short exact sequence of presheaves of groups

$$0 \to A \xrightarrow{\mathfrak{i}} \mathcal{E} \xrightarrow{\mathfrak{f}} \mathrm{G} \to 1$$

in which A is of abelian groups and the induced G-module structure on A is the given one.

Isomorphism classes of singular extensions of a presheaf of groups G by a G-module A correspond bijectively to the elements of $H^2_s(\mathrm{G}, A)$.

With respect to prestacks, we see that for a given G-module $A : C^{op} \to \mathbf{Ab}$ and a simple 3-cocycle $h \in Z^3_s(\mathrm{G}, A)$, the associated prestack is easily recognized as a prestack of categorical groups (which is full 2-subcategory of the 2-category of monoidal abelian groupoids)

$$\mathfrak{P}(\mathrm{G}, A, h) : \mathrm{C}^{op} \to \text{Categorical Groups}.$$

Therefore prestacks of categorical groups over C (or C-fibred categories on categorical groups) are classified by triples (G, A, c) where G is a presheaf of groups over C, A is a G-module, and $c \in H^3_s(\mathrm{G}, A)$.

Thanks !!