$\mathcal{D}$–modules and Logarithmic Comparison Theorems

Categorical, homological and computational methods in Algebra

Celebrating the 70th birthday of Manuel Ladra

Santiago de Compostela, June 24-26, 2026

F.J. Castro Jiménez. Universidad de Sevilla

imus instituto de matemáticas universidad de sevilla

EXCELENCIA MARÍA DE MAEZTU

INSTITUTO ANDALUZ DE MATEMÁTICAS

Categorical, homological, and computational methods in Algebra

A conference honouring the mathematical contribution of Manuel Ladra, celebrating his 70th birthday

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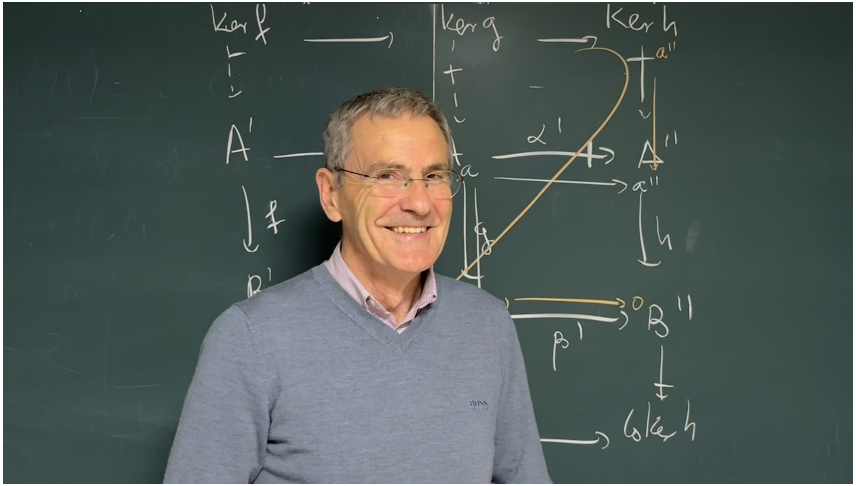

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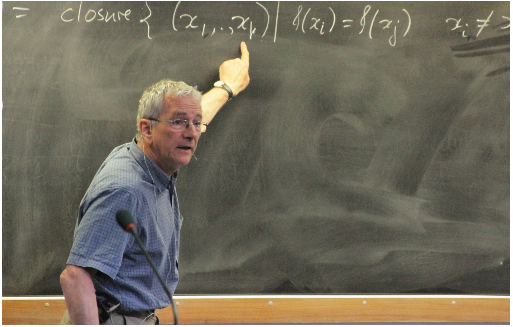
Main reference

This talk is based on the paper:

Logarithmic Comparison Theorems
D. Mond, L. Narváez-Macarro, and FJCJ

Chapter 12 of Handbook of Geometry and Topology of Singularities IV. Cisneros-Molina, J. L., Tráng, L. D., & Seade, J. (Eds.). (2023).
Springer Nature Switzerland AG



David Mond. Crédito de la foto: Reinaldo Mizutani

Luis Narváez Macarro

Grothendieck's Comparison Theorem, 1966

Assume X is a Stein manifold. For $0 \leq q \leq \dim X$:

$$h^q\big(\Gamma(X, \Omega^\bullet_X(*D))\big) \xrightarrow[\simeq]{\mathrm{dR}_q} H^q(X \setminus D, \mathbb{C})$$

Setup

X a (Stein) complex manifold

$D \subset X$ a hypersurface
$\Omega^\bullet_X(*D)$ meromorphic differential forms with poles in D

dR_q the de Rham morphism

$h^q(\Gamma(X, \Omega^\bullet_X(*D))) \equiv$ (co)homology of the complex of global sections of meromorphic differential forms

$H^q(X \setminus D, \mathbb{C}) \equiv$ singular cohomology

The de Rham morphism

$$\mathrm{dR}_q : h^q\big(\Gamma(X, \Omega_X^\bullet(*D))\big) \longrightarrow H^q(X \setminus D, \mathbb{C})$$

$$\mathrm{dR}_q([\omega])([\gamma]) := \int_\gamma \omega \in \mathbb{C}$$

$\omega \in \Omega_X^q(*D)$ and $[\gamma] \in H_q(X \setminus D, \mathbb{C})$

If D is a normal crossing divisor (NCD), this comparison result was proved by Atiyah and Hodge (1955)

Grothendieck's proof (1966) reduces the general case to the case of a NCD by using Hironaka's resolution of singularities (1964)

The de Rham morphism: special case, Brieskorn 1971

When $D \subset \mathbb{C}^n$ is a hyperplane arrangement defined by

$$h_1 \cdots h_r = 0,$$

$B^\bullet(D)$ the Brieskorn complex (the subcomplex of $\Gamma(\mathbb{C}^n, \Omega^\bullet_{\mathbb{C}^n}(*D))$ generated by the forms dh_i/h_i)

Brieskorn proved (1971) that

$$\mathrm{dR}_\bullet : B^\bullet(D) \xrightarrow{\simeq} H^\bullet(\mathbb{C}^n \setminus D, \mathbb{C})$$

is an isomorphism.

Grothendieck's Comparison Theorem

Grothendieck's Comparison result *for Stein manifolds* can be generalized

It is valid on any complex manifold X and any divisor $D \subset X$

The natural "de Rham" morphism

$$\Omega_X^\bullet(*D) \longrightarrow \mathbb{R}j_*\mathbb{C}_{X\setminus D}$$

is an isomorphism in *the derived category of sheaves of $\mathbb{C}_X$-vector spaces*. Here $j : X \setminus D \hookrightarrow X$ is the inclusion.

Since each cohomology class in $H^\bullet(X \setminus D; \mathbb{C})$ is represented by a meromorphic form, **what can be said about the order of its pole along D?**

Logarithmic differential forms (K. Saito, 1980)

$$i_D (= i_D^\bullet) : \Omega_X^\bullet(\log D) \longrightarrow \Omega_X^\bullet(*D)$$

$\omega \in \Omega_X^\bullet(*D)$ is logarithmic along $D \subset X$ if both $h\omega$ and $hd\omega$ are regular (h is a local equation for D).

Notice that $B^\bullet(D) \hookrightarrow \Omega_{\mathbb{C}^n}^\bullet(\log D)$, $D \subset \mathbb{C}^n$ a hyperplane arrangement

Deligne proved (1970) that if $D \subset X$ is a normal crossing divisor (NCD) then i_D is a quasi-isomorphim

Logarithmic Comparison Theorem

Definition: If the morphism $i_D : \Omega^\bullet(\log D) \hookrightarrow \Omega^\bullet(*D)$ is a quasi-isomorphism, we say that the divisor D satisfies the Logarithmic Comparison Theorem (LCT) (the LCT holds for D)

By Grothendieck Comparison Theorem, the LCT holds for D if and only if the "de Rham" morphism $\Omega^\bullet(\log D) \to \mathbb{R}j_*\mathbb{C}_{X\setminus D}$ is a quasi-isomorphism

Theorem [Mond-Narváez-FJCJ, 1996]: If $D \subset X$ is a *locally quasihomogeneous free divisor*, then the de Rham morphism $\Omega^\bullet(\log D) \to \mathbb{R}j_*\mathbb{C}_{X\setminus D}$ is a quasi-isomorphism.

LQH free divisors

Definition: (K. Saito, 1980) D is a free divisor if $\Omega^1(\log D)$ is locally free over $\mathcal{O}_X$

D is locally quasihomogeneous (LQH) if at any point $p \in D$, and in suitable coordinates, (D, p) is defined by a weighted homogeneous polynomial (all weights > 0)

LQH free divisors

EXAMPLES OF FREE DIVISORS: Smooth divisor, plane curve, NCD (and *free* hyperplane arrangement), discriminant of a stable map in Mather *nice dimensions*

EXAMPLES OF LQH DIVISORS: Smooth divisor, quasi-homogeneous plane curve, NCD, hyperplane arrangement, discriminant of a stable map in Mather *nice dimensions*

EXAMPLES OF NON FREE DIVISORS:

- $D \equiv (x_1 \cdots x_n (x_1 + \cdots + x_n) = 0) \subset \mathbb{C}^n$, $n \geq 3$
- $D \equiv (x_1^d + \cdots + x_n^d = 0) \subset \mathbb{C}^n$, $n \geq 3, d \geq 2$

LQH?

- LCT $\Rightarrow$ Local quasihomogeneity for plane curves (Calderón-Moreno, Mond, Narváez-Macarro, FJCJ, 2002)

- LCT $\Rightarrow$ Strong Euler homogeneity for
  - free surfaces in 3-space (Granger, Schulze, 2006)
  - free hypersurfaces in 4-space (A. del Valle, 2025).

- For free divisors in higher dimensions, local weak quasihomogeneity is sufficient (Gago-Vargas, Hartillo-Hermoso, Ucha-Enríquez, and FJCJ (2007), and (Narváez-Macarro, 2008).

LCT for q.h. isolated singularities, Holland and Mond, 1998

Theorem [Holland-Mond, 1998]: Suppose $n = \dim X \geq 3$, $(D, p) \subset (X, p)$ is a quasihomogeneous isolated singularity with integer weights $w_1 > 0, \ldots, w_n > 0$.
Let $f \in \mathcal{O}_{X,p}$ be a reduced equation of D at p with weight $r > 0$.
Then LCT holds for (D, p) if and only if $(\mathcal{O}_{X,p}/J_f)_{ri-\sum_j w_j} = (0)$ for $1 \leq i \leq n-2$, where J_f is the Jacobian ideal.

Conjecture of Terao, 1977

H. Terao conjectured that LCT always holds for hyperplane arrangements

Partial results by Wiens and Yuzvinsky, 1997.

Terao's conjecture proved by D. Bath, 2024.

The sheaf $\mathcal{D}_X$

X: complex manifold

$\mathcal{O}_X$: sheaf of holomorphic functions on X

$\mathcal{D}_X$: sheaf of linear holomorphic differential operators on X

A local section P of $\mathcal{D}_X$ can be written as a finite sum

$$P = \sum_{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n} c_\alpha \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$$

where c_α is a local section of $\mathcal{O}_X$

GCT revisited

Mebkhout (1979) interprets Grothendieck's comparison theorem

$$\Omega_X^\bullet(*D) \xrightarrow{\simeq} \mathbb{R}j_*\mathbb{C}_{X\setminus D}$$

for any divisor $D \subset X$

as the *regularity* of the (holonomic) $\mathcal{D}_X$–module $\mathcal{O}_X$

A $\mathcal{D}$-module Criterion for LCT for free divisors

Theorem [Calderón-Moreno and Narváez-Macarro (2005) and Narváez-Macarro (2007)]
Assume D is free. The LCT holds for D if and only the natural map

$$\varrho : \mathcal{D} \overset{\mathbb{L}}{\otimes}_{\mathcal{V}^D} \mathcal{O}(D) \longrightarrow \mathcal{O}(\star D)$$

is an isomorphism in the derived category of $\mathcal{D}$-modules.

$\mathcal{V}^D \subset \mathcal{D}$: sheaf of logarithmic linear differential operators, with respect to $D \subset X$ (Calderón-Moreno, 1999)

Computability: CM-NM (2005), NM (2007)

(K. Saito, 1980) $\mathrm{Der}(-\log D)$: sheaf of vector fields tangent to D.

$D \subset X$ is free $\Leftrightarrow$ $\mathrm{Der}(-\log D)$ is $\mathcal{O}_X$-locally free.

Let $\{\delta_1, \ldots, \delta_n\}$ be a basis of $\mathrm{Der}(-\log D)_p$, $p \in D$; $\delta_i(f) = \alpha_i f$, $\alpha_i \in \mathcal{O}_{X,p}$.

The map $\varrho_p : \mathcal{D}_p \overset{\mathbb{L}}{\otimes}_{\mathcal{V}_p^D} \mathcal{O}_p(D) \longrightarrow \mathcal{O}_p(\star D)$ is an isomorphism (in the derived category of $\mathcal{D}_p$-modules) if and only if the following properties hold:

(i) The complex $\mathcal{D}_p \overset{\mathbb{L}}{\otimes}_{\mathcal{V}_p^D} \mathcal{O}_p(D)$ is exact in cohomological degrees $\neq 0$, and

(ii) $\mathcal{O}_p(\star D) \simeq \mathcal{D}_p / \mathcal{D}_p \langle \delta_1 + \alpha_1, \ldots, \delta_n + \alpha_n \rangle$.

Quasi-free divisors. Ucha-Enríquez, FJCJ, 2004

Definition: A *quasi-free structure* on (X, D) consists of a locally free sub–$\mathcal{O}_X$–module $\mathcal{L} \subset \mathcal{D}er(-\log D)$ of constant rank n, which is a sub-Lie algebroid of $\mathcal{D}er(-\log D)$, and such that $\operatorname{supp}(\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X)/\mathcal{L}) = D$

free $\Rightarrow$ quasi-free

For any germ of a hypersurface (D, p) with a reduced equation $f \in \mathcal{O}_{X,p}$, $\mathcal{L} := f\,\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,p})$ defines the so-called *trivial* quasi-free structure on (D, p)

Examples of quasi-free structures

The non free divisors

$$(x^2 + y^2 + z^2 = 0) \subset \mathbb{C}^3$$

and

$$(x_1 \cdots x_n (x_1 + \cdots + x_n) = 0) \subset \mathbb{C}^n, \; n \geq 3$$

admit non-trivial quasi-free structures

Examples of quasi-free structures

Let $\mathfrak{X}$ be a generic square matrix of order $n \geq 1$ with entries x_{ij}. Such an $\mathfrak{X}$ is a point in the representation space $\mathrm{Rep}(Q, \mathsf{d})$ of the star quiver: a sink node and n outer nodes,

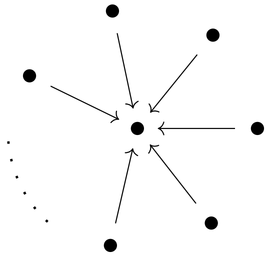


Figure: The star quiver with n outer nodes and one sink.

$$\mathrm{Rep}(Q, \mathsf{d}) = \prod_{i=1}^{n} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}) \simeq \mathrm{Mat}_n(\mathbb{C}).$$

Examples of quasi-free structures

The associated Lie algebra is

$$\mathfrak{g}(Q,\mathrm{d}) \simeq \mathfrak{gl}(n,\mathbb{C}) \times \prod_{i=1}^{n-1} \mathfrak{gl}(1,\mathbb{C})$$

Let $L(Q,\mathrm{d})$ be the Lie algebroid associated to the infinitesimal action. It is generated by

$$\delta_{ij} = \sum_{k=1}^{n} x_{jk}\partial_{ik}, \;\; 1 \leq i,j \leq n$$

Note that $\delta_{ij}(\det \mathfrak{X}) = (\text{Kronecker delta})_i^j \det \mathfrak{X}$
The determinant of the correspondent “the Saito matrix" is $(\det \mathfrak{X})^n$

$L(Q,\mathrm{d})$ is a quasi-free structure on $D \equiv (\det \mathfrak{X} = 0) \subset \mathrm{Rep}(Q,\mathrm{d})$

A $\mathcal{D}$-module Criterion for LCT for quasi-free structures (Mond-Narváez-FJCJ (WIP)

Assume $(D, \mathcal{L})$ is a quasi-free structure. The LCT holds for $(D, \mathcal{L})$ if and only the natural map

$$\varrho : \mathcal{D} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{\mathcal{L}}} (\det \mathcal{L})^* \longrightarrow \mathcal{O}(\star D)$$

is an isomorphism in the derived category of $\mathcal{D}$-modules.

$\mathcal{D}_{\mathcal{L}} \subset \mathcal{D}$: sheaf of logarithmic lineal differential operators, with respect to the quasi-free structure $(D, \mathcal{L})$. Since $\det \mathcal{L}$ is a sub-$\mathcal{D}_{\mathcal{L}}$-module of O_X, the dual $(\det \mathcal{L})^* := \mathcal{H}om_{O_X}(\det \mathcal{L}, O_X)$ is a left $\mathcal{D}_{\mathcal{L}}$-module.

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¡Feliz aniversario e que cumpras moitos máis!

Por xiadas, por calores,
desde qu'amañece o día
dou á terra os meus sudores,
mais canto esa terra cría,
todos ... todo é dos señores.

Rosalía de Castro. Cantares Gallegos,
publicada en *El Museo Universal* (1861).