# Projective crossed modules in semi-abelian categories

Maxime Culot

Université catholique de Louvain

June 2026

Dedicated to Manuel Ladra on the occasion of his 70th birthday

# Why (internal) crossed modules are not Schreier

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## TODAY:

If we replace groups by any semi-abelian category (e.g. groups, Lie algebras over a commutative ring, crossed modules), then we can use the proof of Carrasco, Cegarra and Grandjeán to prove that "crossed modules" are NEVER SCHREIER !

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# What is a crossed module?

A crossed module $(T, G, \partial)$ is given by a morphism of groups $\partial\colon T \to G$ and by a group action of $G$ on $T$ (denoted ${}^{g}t$) satisfying

$$\partial\left({}^{g}t\right) = g\partial(t)g^{-1} \qquad {}^{\partial(t)}t' = tt't^{-1}$$

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A morphism of crossed modules $(f_T, f_G)\colon (T, G, \partial) \to (T', G', \partial')$ is a pair of group morphisms $f_T\colon T \to T'$ and $f_G\colon G \to G'$ such that

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XMod = crossed modules + morphisms of crossed modules.

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To define a group, we need three operations

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$\begin{tikzcd} \mathcal{V} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, bend right, "Fr"'] \end{tikzcd}$ where $U$ forgetful functor and $Fr$ free functor, i.e.

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A Schreier variety = a variety where a subobject of a free object is again free. Example: groups, by Nielsen-Schreier's theorem. Counter-example: XMod [CCRG02].

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Let $A$ and $B$ two crossed modules,

$$A \flat B := \operatorname{Ker}(\langle 1_A, 0 \rangle \colon A + B \to A) \trianglelefteq A + B.$$

We denote by $Fr(X)$ the free group on the set $X$.

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As a result, XMod is not Schreier: we have a projective crossed module which is not free!!

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As in groups:

- Any kernel is an internal crossed module;
- Morphisms of internal crossed modules = morphisms in $\mathcal{C}$ compatible with $\partial$'s and with action cores ($\psi\colon G \diamond T \to T$)

# A special projective crossed module

**Proposition [CCRG02]**

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**Theorem [Cul26]**

Consider a semi-abelian category $\mathcal{C}$. If $P$ is a projective object in $\mathcal{C}$ and if the split extension

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r, tail, "k"] & Q \rtimes_{\psi} P \arrow[r, shift left, "p"] & P \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

is a projective object in the category of split extensions of $P$, then the kernel $k$ is a projective object in $\mathrm{XMod}(\mathcal{C})$.

# Why is this a projective crossed module?

Consider a "surjection" $(f_X, f_A)\colon (X, A, \partial) \to (Q, Q \rtimes_\psi P, k)$ in $\mathsf{XMod}(\mathcal{C})$:

$$\begin{tikzcd}
X \arrow[r, "\partial"] \arrow[d, two heads, "f_X"'] & A \arrow[d, two heads, "f_A"] & \\
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3. With $g$ and $g_X$, define a section of $f_A$: $g_A$ ($Q \rtimes_\psi P$);
4. The pair of sections $(g_X, g_A)$ is a morphism in $\mathrm{XMod}(\mathcal{C})$ ("$\diamond$" characterization - compatible with the actions).

# How to build free crossed modules ?

## Proposition [CCRG02]

The normal subgroup $Fr(X)\flat Fr(X)$ of $Fr(X)+Fr(X)$, where $X$ is a set and $Fr(X)$ is the free group on $X$, is the free crossed module over $X$.

$$\begin{tikzcd} \mathrm{XMod}(\mathrm{Gp}) \arrow[r, shift right=2, "R"'] \arrow[r, phantom, "\perp"] & \mathrm{Gp} \arrow[l, shift right=2, "L"'] \arrow[r, shift right=2, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift right=2, "Fr"'] \end{tikzcd}$$

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where $\mathcal{V}$ is a semi-abelian variety.

## Lemma [Cul26]

A free crossed module in $\mathrm{XMod}(\mathcal{V})$ is given by

$$(Fr(X)\flat Fr(X), Fr(X) + Fr(X), \kappa_{Fr(X),Fr(X)})$$

for some $X \in \mathrm{Set}$.

# $\mathsf{XMod}(\mathcal{V})$ is not a Schreier variety

Any free crossed module in $\mathsf{XMod}(\mathcal{V})$ is given by

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for some $X \in$ Set.

## Corollary [Cul26]

For any semi-abelian non-trivial variety $\mathcal{V}$, the variety XMod($\mathcal{V}$) is not a Schreier variety.

## Sketch of the proof

Consider two different projectives objects $P$ and $Y$ in $\mathcal{V}$, then

$$P \flat Y \overset{\kappa_{P,Y}}{\rightarrowtriangle} P + Y \underset{\iota_1}{\overset{\langle 1_P,0 \rangle}{\rightleftarrows}} P$$

the kernel part is projective in XMod($\mathcal{V}$) but not free since $P \neq Y$.

Crossed modules: definition and classical results

Internal crossed modules: new general results

A glance of homological algebra

# What are the ingredients to define non-additive left derived functors?

Left derived functors are defined when

- $\mathcal{C}$ and $\mathcal{E}$ abelian categories (as abelian groups, not groups);
- $\mathcal{C}$ with enough projectives (every variety);
- $F\colon \mathcal{C} \to \mathcal{E}$ additive functor.

In $\mathcal{C}$ and $\mathcal{E}$, the hom-sets are abelian groups and $F$ preserves this structure.

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- $\mathcal{C}$ with enough projectives (every variety);
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In $\mathcal{C}$ and $\mathcal{E}$, the hom-sets are abelian groups and $F$ preserves this structure.

Non-additive left derived functors [CRVdL25], we need

- $\mathcal{C}$ and $\mathcal{E}$ semi-abelian categories (groups, XMod, ...);
- $\mathcal{C}$ satisfies condition (P) [CRVdL25];
- $\mathcal{C}$ with enough projectives (every variety);
- $F\colon \mathcal{C} \to \mathcal{E}$ protoadditive functor [EG10] + "conditions"

# What is the condition (P) ?

In a semi-abelian category, condition (P): consider

$$
\begin{tikzcd}
K \arrow[r, tail, "k"] & X \arrow[r, shift left, two heads, "f"] & Y \arrow[l, shift left, "s"]
\end{tikzcd}
$$

$f \circ s = 1_Y$ and $K$ is the kernel of $f$, if $X$ is projective then $K$ is projective.

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Trivial examples: abelian categories and Schreier varieties;

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Non-trivial examples: Lie algebras over a commutative ring, XMod, XMod($\mathcal{V}$) ($\mathcal{V}$ semi-abelian variety satisfying $(\mathrm{P})$).

The proof of "if $\mathcal{V}$ satisfies $(\mathrm{P})$ then XMod($\mathcal{V}$) as well" is based on the construction of the special projective crossed module and of the free crossed modules.

# First example of a non-additive left derived functor with $\mathsf{XMod}(\mathcal{V})$

We can compute the non-additive left derived functor of $\pi_0 \colon \mathsf{XMod}(\mathcal{V}) \to \mathcal{V}$ where

- $\mathcal{V}$ is a semi-abelian variety satisfying $(\mathrm{P})$ (e.g. groups, XMod);
- then $\mathsf{XMod}(\mathcal{V})$ is a non-trivial example of semi-abelian varieties satisfying $(\mathrm{P})$;
- $\pi_0(X, A, \partial \colon T \to G) := \mathsf{Coker}(\partial \colon T \to G)$.

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- the construction of the free crossed modules;

Leading to: for any semi-abelian non-trivial variety $\mathcal{V}$, XMod($\mathcal{V}$) is not a Schreier variety;

My motivation for this topic (homological algebra): we can compute the non-additive left derived functors of $\pi_0 \colon \mathrm{XMod}(\mathcal{V}) \to \mathcal{V}$ (where $\mathcal{V}$ is a semi-abelian variety satisfying $(\mathrm{P})$).

Happy birthday Manuel Ladra !

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