A magic square of Lie superalgebras via tensor categories

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Outline

1. J-ternary algebras
2. Structurable algebras
3. From algebras to superalgebras via tensor categories
4. From J-ternary algebras to Lie superalgebras in characteristic 3
5. A magic square of Lie superalgebras

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J-ternary algebras

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Definition (Hein 1975)

Let $\mathcal{T}$ be a triple system with ternary multiplication xyz. Then $\mathcal{T}$ is said to be a **J-ternary algebra** if the following conditions hold:

$$xy(uvz) = (xyu)vz + u(yxv)z + uv(xyz),$$
$$xyz - zyx = zxy - xzy,$$

for all $x, y, z, u, v \in \mathcal{T}$.

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Assume $\langle . \mid . \rangle : \mathcal{T} \times \mathcal{T} \to \mathcal{J}$ is a skew-symmetric bilinear map and $(.,.,.) : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a trilinear product on $\mathcal{T}$.

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Assume $\langle . \mid . \rangle : \mathcal{T} \times \mathcal{T} \to \mathcal{J}$ is a skew-symmetric bilinear map and $(., ., .) : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a trilinear product on $\mathcal{T}$. Then $\mathcal{T}$ is called a **$\mathcal{J}$-ternary algebra** if the following axioms hold for any $a \in \mathcal{J}$ and $x, y, z, w, v \in \mathcal{T}$:

$$a \cdot \langle x \mid y \rangle = \frac{1}{2}(\langle a \bullet x \mid y \rangle + \langle x \mid a \bullet y \rangle),$$

$$a \bullet (x, y, z) = (a \bullet x, y, z) - (x, a \bullet y, z) + (x, y, a \bullet z),$$

$$(x, y, z) = (z, y, x) - \langle x \mid z \rangle \bullet y,$$

$$(x, y, z) = (y, x, z) + \langle x \mid y \rangle \bullet z,$$

$$\langle (x, y, z) \mid w \rangle + \langle z \mid (x, y, w) \rangle = \langle x \mid \langle z \mid w \rangle \bullet y \rangle,$$

$$(x, y, (z, w, v)) = ((x, y, z), w, v) + (z, (y, x, w), v) + (z, w, (x, y, v)).$$

J-ternary algebras

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Conversely, let $\mathcal{T}$ be a J-ternary algebra, then

- $\mathcal{J} := \mathbb{F}\mathrm{id} + K(\mathcal{T}, \mathcal{T})$ *is a Jordan subalgebra of* $\mathrm{End}_{\mathbb{F}}(\mathcal{T})^{(+)}$*, where* $K(x,y)z = xzy - yzx$ *for* $x, y \in \mathcal{T}$ *and* $K(\mathcal{T}, \mathcal{T})$ *is the linear span of the* $K(x,y)$*'s,*
- $\mathcal{T}$ *becomes a* $\mathcal{J}$*-ternary algebra with the natural action of* J *on* $\mathcal{T}$*:* $a \bullet x = a(x)$*, and the multilinear maps:*

$$(.,.,.) : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}, \quad (x,y,z) = xyz,$$
$$\langle . \mid . \rangle : \mathcal{T} \times \mathcal{T} \to \mathcal{J}, \quad \langle x \mid y \rangle = -K(x,y).$$

Prototypical example

Let $(\mathcal{A},*)$ be a unital associative algebra with involution and let $\mathcal{T}$ be a left $\mathcal{A}$-module endowed with a skew-hermitian form $\mathsf{h} : \mathcal{T} \times \mathcal{T} \to \mathcal{A}$. Then $\mathcal{T}$ is a $\mathcal{J}$-ternary algebra, where $\mathcal{J} = \mathcal{H}(\mathcal{A},*)$ is the Jordan algebra of symmetric elements of $\mathcal{A}$ relative to the involution $*$, with the following operations:

- $a \cdot b = \frac{1}{2}(ab + ba)$ for any $a, b \in \mathcal{J} = \mathcal{H}(\mathcal{A},*)$,
- $a \bullet x = ax$ for any $a \in \mathcal{J}$ and $x \in \mathcal{T}$,
- $\langle x \mid y \rangle = \mathsf{h}(x,y) - \mathsf{h}(y,x)$ for $x, y \in \mathcal{T}$, and
- $(x,y,z) = \mathsf{h}(x,y)z + \mathsf{h}(z,x)y + \mathsf{h}(z,y)x$ for $x, y, z \in \mathcal{T}$.

J-ternary algebras and 5-graded Lie algebras

Theorem

Let $\mathcal{T}$ be a J-ternary algebra and consider the Jordan algebra $\mathcal{J} = \mathbb{F}\mathrm{id} + K(\mathcal{T}, \mathcal{T})$, then the direct sum

$$\mathcal{L}(\mathcal{T}) := (\mathfrak{sl}_2 \otimes \mathcal{J}) \oplus (\mathbb{F}^2 \otimes \mathcal{T}) \oplus S(\mathcal{T}, \mathcal{T})$$

is a 5-graded Lie algebra with a natural Lie bracket, where

$$S(\mathcal{T}, \mathcal{T}) = \textit{span}\,\{S(x, y) = L(x, y) + L(y, x) \mid x, y \in \mathcal{T}\} \leq \mathfrak{gl}(\mathcal{T}),$$

$$\mathcal{L}(\mathcal{T})_2 = E \otimes \mathcal{J}, \quad \mathcal{L}(\mathcal{T})_0 = (H \otimes \mathcal{J}) \oplus S(\mathcal{T}, \mathcal{T}), \quad \mathcal{L}(\mathcal{T})_{-2} = F \otimes \mathcal{J},$$

$$\mathcal{L}(\mathcal{T})_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathcal{T}, \quad \mathcal{L}(\mathcal{T})_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \mathcal{T},$$

J-ternary algebras and 5-graded Lie algebras

Conversely, let $\mathcal{L}$ be a 5-graded Lie algebra:

$$\mathcal{L} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2,$$

and assume that there are elements $E \in \mathcal{L}_2$ and $F \in \mathcal{L}_{-2}$, such that span $\{E, F, H = [E, F]\}$ is a subalgebra isomorphic to $\mathfrak{sl}_2$, with $\mathcal{L}_i = \{X \in \mathcal{L} \mid [H, X] = iX\}$, for $i = -2, -1, 0, 1, 2$. Then $\mathcal{L}_1$ is a J-ternary algebra with the triple product

$$(X, Y, Z) = \frac{1}{2}[[X, [F, Y]], Z]$$

for $X, Y, Z \in \mathcal{L}_1$.

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Structurable algebras

Definition (Allison 1978, Allison and Faulkner 1993)

A unital algebra with involution $(\mathcal{A}, -)$ is a **structurable algebra** if the following two conditions hold:

$$[V_{a,b}, V_{c,d}] = V_{V_{a,b}(c),d} - V_{c,V_{b,a}(d)}$$
$$(a - \overline{a}, b, c) = (b, \overline{a} - a, c)$$

for any $a, b, c, d \in \mathcal{A}$, where $(a, b, c) := (ab)c - a(bc)$ is the associator of a, b, c, and $V_{a,b}(c) = (a\bar{b})c + (c\bar{b})a - (c\overline{a})b$.

Structurable algebras and 5-graded Lie algebras

Theorem (Allison 79 in case the characteristic is $\neq 2, 3$)

Let $(\mathcal{A}, -)$ be a structurable algebra. Then the vector space

$$\mathcal{K}(\mathcal{A}, -) = \mathcal{N}^{\sim} \oplus \mathfrak{instrl}(\mathcal{A}, -) \oplus \mathcal{N},$$

where $\mathcal{N} = \mathcal{A} \times \mathcal{S} = \{(x, s) \mid x \in \mathcal{A},\ s \in \mathcal{S}\}$, $\mathcal{N}^{\sim}$ is a copy of $\mathcal{N}$, is a 5-graded Lie algebra with

$$\mathcal{K}_2 = \{(0, s) \mid s \in \mathcal{S}\} \quad \mathcal{K}_1 = \{(x, 0) \mid x \in \mathcal{A}\}, \quad \mathcal{K}_0 = \mathfrak{instrl}(\mathcal{A}, -),$$
$$\mathcal{K}_{-1} = \{(x, 0)^{\sim} \mid x \in \mathcal{A}\}, \quad \mathcal{K}_{-2} = \{(0, s)^{\sim} \mid s \in \mathcal{S}\}.$$

Structurable algebras and 5-graded Lie algebras

The bracket is given by imposing that $\mathfrak{instrl}(\mathcal{A},-)=\operatorname{span}\{V_{x,y}\mid x,y\in\mathcal{A}\}$ is a subalgebra, together with:

$$\begin{aligned}
[T,(x,s)] &= \big(T(x),T^{\delta}(s)\big),\\
[T,(x,s)^{\sim}] &= \big(T^{\varepsilon}(x),T^{\varepsilon\delta}(s)\big)^{\sim},\\
[(x,s),(y,t)] &= (0,x\overline{y}-y\overline{x}),\\
[(x,s)^{\sim},(y,t)^{\sim}] &= (0,x\overline{y}-y\overline{x})^{\sim},\\
[(x,s),(y,t)^{\sim}] &= -(tx,0)^{\sim}+\big(V_{x,y}+L_sL_t\big)+(sy,0),
\end{aligned}$$

for $x,y\in\mathcal{A}$, $s,t\in\mathcal{S}$, $T\in\mathfrak{instrl}(\mathcal{A},-)$, where $V^{\varepsilon}_{x,y}=-V_{y,x}$ and $T^{\delta}=T+R_{\overline{T(1)}}$.

Structurable algebras and J-ternary algebras

Theorem

Let $(\mathcal{A}, -)$ be a structurable algebra and let s be a skew-symmetric element such that the left multiplication L_s is bijective. Then $\mathcal{A}$ is a J-ternary algebra with the triple product

$$(x, y, z) = V_{x,sy}(z)$$

for $x, y, z \in \mathcal{A}$.

Structurable algebras and J-ternary algebras

Moreover, $\mathcal{S} = \{x \in \mathcal{A} \mid \overline{x} = -x\}$ is a Jordan algebra with multiplication

$$a \cdot b = \frac{1}{2}\big(a(sb) + b(sa)\big),$$

for $a, b \in \mathcal{S}$; $\mathcal{A}$ is a special Jordan $\mathcal{S}$-module with

$$a \bullet x = a(sx),$$

for $a \in \mathcal{S}$ and $x \in \mathcal{A}$; and $\mathcal{A}$ is an $\mathcal{S}$-ternary algebra with the triple product above and with

$$\langle x \mid y \rangle = y\overline{x} - x\overline{y},$$

for $x, y \in \mathcal{A}$.

Example

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- $\mathcal{W}$ is endowed with a hermitian form $\mathsf{h} : \mathcal{W} \times \mathcal{W} \to \mathcal{E}$, so that $\mathsf{h}(y, x) = \overline{\mathsf{h}(x, y)}$ and $\mathsf{h}(e \circ x, y) = e\mathsf{h}(x, y)$, for any $x, y \in \mathcal{W}$ and $e \in \mathcal{E}$,

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- the involution on $\mathcal{A}$ is defined by $\overline{e + x} = \overline{e} + x$, for $e \in \mathcal{E}$ and $x \in \mathcal{W}$.

Example

- The multiplication in $\mathcal{A}$ is defined as follows:

$$(e_1 + x_1)(e_2 + x_2) = \big(e_1 e_2 + \mathsf{h}(x_2, x_1)\big) + \big(\overline{e_1} \circ x_2 + e_2 \circ x_1\big)$$

for $e_1, e_2 \in \mathcal{E}$ and $x_1, x_2 \in \mathcal{W}$.

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Take an element $s \in \mathcal{S} = \{x \in \mathcal{A} \mid \overline{x} = -x\} = \{a \in \mathcal{E} \mid \overline{a} = -a\}$, with L_s invertible. Then $\mathcal{S}$ is a Jordan algebra with $a \cdot b = \frac{1}{2}(asb + bsa)$ for $a, b \in \mathcal{S}$ ($\mathcal{E}$ is associative, so there is no need of parentheses), and $\mathcal{A}$ is an $\mathcal{S}$-ternary algebra.

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Not surprisingly, as an $\mathcal{S}$-ternary algebra, $\mathcal{A}$ lies in the setting of our *prototypical example*.

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- Lie if it is anticommutative: $\mu \circ c_{\mathcal{A},\mathcal{A}} = -\mu$, and

$$\mu \circ (\mu \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}} + c_{\mathcal{A}\otimes\mathcal{A},\mathcal{A}} + c_{\mathcal{A},\mathcal{A}\otimes\mathcal{A}}) = 0,$$

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Superalgebras are algebras in $\mathsf{sVec}_{\mathbb{F}}$.

Traces in symmetric tensor categories

Given a morphism $f \in \mathrm{End}_{\mathfrak{C}}(X)$ in a symmetric tensor category, its trace $\mathrm{tr}_X(f)$ is the following element in $\mathrm{End}_{\mathfrak{C}}(\mathbf{1}) \simeq \mathbb{F}$:

$$\mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{f \otimes \mathrm{id}_{X^*}} X \otimes X^* \xrightarrow{c_{X,X^*}} X^* \otimes X \xrightarrow{\mathrm{ev}_{\mathrm{X}}} \mathbf{1}$$

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The dimension of an object X is $\dim_{\mathfrak{C}}(X) := \mathrm{tr}_X(\mathrm{id}_X)$.

Negligible morphisms

A morphism $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ in a symmetric tensor category is said to be negligible if

$$\mathrm{tr}_Y(f \circ g) = 0 \quad \text{for all} \quad g \in \mathrm{Hom}_{\mathfrak{C}}(Y, X).$$

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The subspaces $\mathcal{N}(X, Y)$ form a tensor ideal.

Semisimplification of a symmetric tensor category

This means that we can define a new category $\mathfrak{C}^{ss}$ with the same objects as $\mathfrak{C}$, but with morphisms given by the quotient with the subspace of negligible morphisms:

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The natural functor $S : \mathfrak{C} \to \mathfrak{C}^{ss}$ which is the identity on objects, and sends any morphism f to its class $[f]$ modulo negligible morphisms is a braided, monoidal, $\mathbb{F}$-linear functor.

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Any indecomposable object in $\mathfrak{C}$ with $\dim_{\mathfrak{C}}(X) = 0$ becomes isomorphic to the zero object in $\mathfrak{C}^{ss}$.

Verlinde category

Let $\mathbb{F}$ be a field of characteristic $p > 0$. The affine group scheme $\boldsymbol{\alpha}_p$ assigns to any unital, commutative, associative ring the additive group

$$\boldsymbol{\alpha}_p(R) = \{r \in R \mid r^p = 0\}.$$

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Its representing object is the Hopf algebra $\mathbb{F}[X]/(X^p)$, and hence a representation of $\boldsymbol{\alpha}_p$ is nothing else but a vector space endowed with a nilpotent endomorphism δ satisfying $\delta^p = 0$.

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Definition

The semisimplification of Rep $\boldsymbol{\alpha}_p$ is called the **Verlinde category** Ver_p.

Semisimplification of Rep $\boldsymbol{\alpha}_p$

- There are, up to isomorphism, p indecomposable objects in Rep $\boldsymbol{\alpha}_p$: $L_1, \ldots, L_p$, where L_i denotes the indecomposable object of dimension i, $1 \leq i \leq p$.

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- These indecomposable objects $L_1, \ldots, L_{p-1}$ become irreducible in Ver$_p$, while L_p is a zero object.
- Assuming $p > 2$, the full tensor subcategory of Ver$_p$ generated by L_1 and L_{p-1} is equivalent to the symmetric tensor category sVec of vector superspaces (over our ground field).

Semisimplification of Rep $\boldsymbol{\alpha}_p$

Therefore, given a Lie algebra $\mathcal{L}$ over $\mathbb{F}$ endowed with a nilpotent derivation $\delta \in \mathfrak{Der}(\mathcal{L})$ such that $\delta^p = 0$, that is, given a Lie algebra $\mathcal{L}$ in Rep $\boldsymbol{\alpha}_p$, we obtain, by considering the Lie bracket modulo negligible homomorphisms, an 'operadic' Lie algebra in Ver_p.

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Moreover, this ‘operadic’ Lie algebra in Ver$_p$ contains a subalgebra, which lies in the subcategory above, and hence it is a Lie algebra in sVec. Note that if $p = 3$, the tensor categories Ver$_3$ and sVec are equivalent, because L_3 is a zero object in Ver$_3$.

Recipe to convert Lie algebras into superalgebras

Theorem

Let $\mathcal{L}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$, endowed with a nilpotent derivation $\delta \in \mathfrak{Der}(\mathcal{L})$ such that $\delta^p = 0$. Decompose $\mathcal{L}$ in a sum of indecomposable modules for the action of δ, so that

$$\mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_p,$$

where $\mathcal{L}_i$ is a sum of indecomposable modules of dimension i for the action of δ, $1 \leq i \leq p$. Write $\mathcal{L}_{\bar{0}} = \mathcal{L}_1$ and choose a subspace $\mathcal{L}_{\bar{1}}$ of $\mathcal{L}_{p-1}$ complementing $\delta(\mathcal{L}_{p-1})$: $\mathcal{L}_{p-1} = \mathcal{L}_{\bar{1}} \oplus \delta(\mathcal{L}_{p-1})$.

Recipe to convert Lie algebras into superalgebras

Define a bracket on $\mathcal{L}^{\mathsf{ss}} := \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ as follows:

$$\begin{aligned}
[x_{\bar{0}}, y_{\bar{0}}] &= \mathrm{proj}_{\mathcal{L}_{\bar{0}}}([x_{\bar{0}}, y_{\bar{0}}]) \\
[x_{\bar{0}}, y_{\bar{1}}] &= \mathrm{proj}_{\mathcal{L}_{\bar{1}}}([x_{\bar{0}}, y_{\bar{1}}]) \\
[x_{\bar{1}}, y_{\bar{0}}] &= \mathrm{proj}_{\mathcal{L}_{\bar{1}}}([x_{\bar{1}}, y_{\bar{0}}]) \\
[x_{\bar{1}}, y_{\bar{1}}] &= \mathrm{proj}_{\mathcal{L}_{\bar{0}}}([x_{\bar{1}}, \delta^{p-2}(y_{\bar{1}})])
\end{aligned}$$

for all $x_{\bar{0}}, y_{\bar{0}} \in \mathcal{L}_{\bar{0}}$ and $x_{\bar{1}}, y_{\bar{1}} \in \mathcal{L}_{\bar{1}}$, with projections relative to the decomposition $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_{p-2} \oplus \mathcal{L}_{\bar{1}} \oplus \delta(\mathcal{L}_{p-1}) \oplus \mathcal{L}_p$.

Then $\mathcal{L}^{\mathsf{ss}}$, with this bracket, is an operadic Lie algebra in sVec, which is isomorphic to a subalgebra of the operadic Lie algebra in Ver_p obtained from the Lie algebra $\mathcal{L}$ in Rep $\boldsymbol{\alpha}_p$.

A magic square of Lie superalgebras via tensor categories

1. J-ternary algebras
2. Structurable algebras
3. From algebras to superalgebras via tensor categories
4. From J-ternary algebras to Lie superalgebras in characteristic 3
5. A magic square of Lie superalgebras

From J-ternary algebras to Lie superalgebras in characteristic 3

Corollary

Let $\mathcal{T}$ be a J-ternary algebra over a field $\mathbb{F}$ of characteristic 3. Then the $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathcal{L}^{\mathsf{ss}}(\mathcal{T})$ with

$$\mathcal{L}^{\mathsf{ss}}(\mathcal{T})_{\bar{0}} = S(\mathcal{T}, \mathcal{T}), \quad \mathcal{L}^{\mathsf{ss}}(\mathcal{T})_{\bar{1}} = \mathcal{T},$$

is a weak Lie superalgebra with the bracket given by the usual bracket in $S(\mathcal{T}, \mathcal{T}) \leq \mathrm{End}_{\mathbb{F}}(\mathcal{T})$, by $[d, x] = d(x)$ for $d \in S(\mathcal{T}, \mathcal{T})$ and $x \in \mathcal{T}$, and by

$$[x, y] = S(x, y)$$

for $x, y \in \mathcal{T} = \mathcal{L}^{\mathsf{ss}}(\mathcal{T})_{\bar{1}}$.

From J-ternary algebras to Lie superalgebras in characteristic 3

Sketch of Proof

The Lie algebra $\mathcal{L}(\mathcal{T}) = (\mathfrak{sl}_2 \otimes \mathcal{J}) \oplus (\mathbb{F}^2 \otimes \mathcal{T}) \oplus S(\mathcal{T},\mathcal{T})$ is endowed with the nilpotent derivation $\delta = \mathrm{ad}_{F\otimes \mathrm{id}}$ ($\delta^3 = 0$) and:

- $\mathfrak{sl}_2 \otimes \mathcal{J}$ is a sum of indecomposable modules (for the action of δ) of dimension 3, that become trivial in $\mathsf{Ver}_3 \cong \mathsf{sVec}$,
- $\mathbb{F}^2 \otimes \mathcal{T}$ is a sum of indecomposable modules of dimension 2,
- $S(\mathcal{T},\mathcal{T})$ is a sum of indecomposable modules of dimension 1.

Now it is enough to apply our recipe.

J-ternary algebras in our prototypical example give classical Lie superalgebras

Theorem

Let $(\mathcal{A}, -)$ be a finite-dimensional simple algebra with involution (i.e., there is no proper ideal closed under the involution!) over an algebraically closed field $\mathbb{F}$ of characteristic 3. Let $\mathcal{T}$ be a left $\mathcal{A}$-module and let $\mathsf{h} : \mathcal{T} \times \mathcal{T} \to \mathcal{A}$ be a nondegenerate skew-hermitian form. Then $\mathcal{T}$ is a J-ternary algebra with triple product

$$(x, y, z) = \mathsf{h}(x, y)z + \mathsf{h}(z, x)y + \mathsf{h}(z, y)x$$

for $x, y, z \in \mathcal{T}$.

J-ternary algebras in our prototypical example give classical Lie superalgebras

Let $\mathcal{L}^{\mathsf{ss}}(\mathcal{T})$ be the associated operadic Lie superalgebra. Then $\mathcal{L}^{\mathsf{ss}}(\mathcal{T})$ is a Lie superalgebra and the following conditions hold:

- If the involution $-$ on $\mathcal{A}$ is of the first kind, then there are $n, m \in \mathbb{Z}_{>0}$, with m even, such that $\mathcal{L}^{\mathsf{ss}}(\mathcal{T})$ is isomorphic to the orthosymplectic Lie superalgebra $\mathfrak{osp}(n|m)$.
- If the involution $-$ on $\mathcal{A}$ is of the second kind, then there are $n, m \in \mathbb{Z}_{>0}$ such that $\mathcal{L}^{\mathsf{ss}}(\mathcal{T})$ is isomorphic to the projective special linear Lie superalgebra $\mathfrak{psl}(n|m)$.

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Lie superalgebras from simple structurable algebras

There is no known classification of the simple J-ternary algebras in characteristic 3, but the known simple J-ternary algebras in characteristic $\neq 2, 3$ have counterparts in characteristic 3.

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We are left with two families of simple structurable algebras:

- Those of skew-dimension one, in which case the associated Lie superalgebras $\mathcal{L}^{\mathsf{ss}}(\mathcal{T})$ are known [E. 2006]: the Brown superalgebra $\mathfrak{br}(2; 3)$, and the Lie superalgebras $\mathfrak{g}(r, 6)$, for $r = 1, 2, 4, 8$, in the 'supermagic square' [Cunha, E. 2007].

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- The tensor products of an octonion algebra and an arbitrary unital composition algebra.

Missing case

Theorem

Let $(\mathcal{A},-)$ *be the structurable algebra obtained as the tensor product of the Cayley algebra* $\mathcal{C}_1$ *and a unital composition algebra* $\mathcal{C}_2$ *over an algebraically closed field of characteristic* 3*. Let* s *be a skew-symmetric element such that* L_s *is invertible, and let* $\mathcal{L}^{\mathsf{ss}}(\mathcal{A})$ *be the Lie superalgebra attached to the* J*-ternary algebra defined on* $\mathcal{A}$ *. Then the Lie superalgebra* $\mathcal{L}^{\mathsf{ss}}(\mathcal{A})$ *is given, up to isomorphism, by the following table:*

$\dim \mathcal{C}_2$	1	2	4	8
$\mathcal{L}^{\mathsf{ss}}(\mathcal{A})$	$\mathfrak{psl}(4\vert 1)$	$\mathfrak{g}(3,3)$	$\mathfrak{el}(5;3)$	$\mathfrak{g}(6,6)$

A magic square of Lie superalgebras

The Lie superalgebras in the previous slide form the last row of the square of the Lie superalgebras obtained using the structurable algebras given by the tensor product of two unital composition algebras:

		$\dim \mathcal{C}_1$			
		1	2	4	8
	1		$\mathfrak{psl}(1\vert 1)$	$\mathfrak{osp}(2\vert 2)$	$\mathfrak{psl}(4\vert 1)$
$\dim \mathcal{C}_2$	2	$\mathfrak{psl}(1\vert 1)$	$\mathfrak{psl}(1\vert 1) \oplus \mathfrak{psl}(1\vert 1)$	$\mathfrak{psl}(2\vert 2)$	$\mathfrak{g}(3,3)$
	4	$\mathfrak{osp}(2\vert 2)$	$\mathfrak{psl}(2\vert 2)$	$\mathfrak{osp}(4\vert 4)$	$\mathfrak{el}(5;3)$
	8	$\mathfrak{psl}(4\vert 1)$	$\mathfrak{g}(3,3)$	$\mathfrak{el}(5;3)$	$\mathfrak{g}(6,6)$

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