Inner ideal structure of orthogonal Lie algebras

Antonio Fernández López (Universidad de Málaga)

Santiago de Compostela, 24-26 June, 2026

We describe the inner ideal structure of the orthogonal Lie algebras, not necessarily finite dimensional, over a field $\mathbb{F}$ of characteristic not 2. Putting particular emphasis on the finite dimensional case, where the Witt index of the bilinear form plays a fundamental role.

- It is amazing the fact that in D_4 the Clifford inner ideal is conjugate (by a diagram automorphism) to the maximal isotropic inner ideal.

- This phenomenon has a Jordan counterpart: the isomorphism (obtained by functoriality) of the Jordan pair $\mathcal{K}_4$ of *skew-symmetric matrices with entries in* $\mathbb{F}$ and the Jordan pair $\mathcal{Q}_6$ of a 6-*dimensional nondegenerate symmetric bilinear form of maximal Witt index* (result known in the classification of the simple finite dimensional Jordan pairs).

Let L be a Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$.

- An *abelian inner ideal* of L is a subspace B of L such that $[[B, L], B] \subseteq B$ and $[B, B] = 0$. Any abelian subalgebra B of L is an abelian inner ideal if and only if $\mathrm{ad}_b^2 L \subseteq B$ for all $b \in B$.

- Let P be a subspace of L such that $[P, P] = 0$ and $\mathrm{ad}_a^2 L = \mathbb{F}a$ for any $a \in P$. Then P is an abelian inner ideal of L called a *point space*.

Let L now be a subalgebra of $R^- : \; [a, b] = ab - ba, \; a, b \in R$, where R is an associative $\mathbb{F}$-algebra.

- An inner ideal U of L is called *isotropic* if $uv = 0$ for all $u, v \in U$.

- Isotropic inner ideal are clearly abelian.

Suppose now that $\mathrm{char}(\mathbb{F}) \neq 2, 3$. A (linear) *Jordan pair* (JP) is a pair of vector spaces $V = (V^+, V^-)$ over $\mathbb{F}$ with trilinear maps

$$\{\cdot\cdot\cdot\} : V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma \ (\sigma = \pm)$$

such that for $x, z, u \in V^\sigma,\ y, v \in V^{-\sigma}$:

(JP1) $\{x, y, z\} = \{z, y, x\}$,

(JP2) $\{u, v, \{x, y, z\}\} =$
$\{\{u, v, x\}, y, z\} - \{x, \{v, u, y\}, z\} + \{x, y, \{u, v, z\}\}$.

- Let $V = (V^+, V^-)$ and $W = (W^+, W^-)$ be JPs over $\mathbb{F}$. A *homomorphism* $\varphi : V \to W$ is a pair of linear maps (φ^+, φ^-), $\varphi^\sigma : V^\sigma \to W^\sigma$, such that

$$\varphi^\sigma\{x, y, z\} = \{\varphi^\sigma(x), \varphi^{-\sigma}(y), \varphi^\sigma(z)\},\ x, z \in V^\sigma,\ y \in V^{-\sigma}.$$

- An *inner ideal* of a JP V is a subspace B of V^σ such that

$$\{B, V^{-\sigma}, B\} \subseteq B.$$

.

Objects: the pairs (L, B), where L is a Lie algebra over $\mathbb{F}$ and B an abelian inner ideal of L,

Morphisms: the pairs (f, f_B), where $f : L \to M$ is a homomorphism of Lie algebras such that $f(B) = C$, and f_B is the restriction of f to B.

- Let $\mathrm{char}(\mathbb{F}) \neq 2, 3$. For any abelian inner ideal B of L, set

$$\mathrm{Ker}_L B = \{z \in L : [B, [B, z]] = 0\}.$$

The pair of vector spaces $\mathrm{Sub}_L B := (B, L/\,\mathrm{Ker}_L B)$, with the triple products

$$\{b, \bar{x}, c\} := [[b, x], c], \quad \{\bar{x}, b, \bar{y}\} := \overline{[[x, b], y]},$$

$b, c \in B,\ x, y \in L$, is a JP called the *subquotient* of L with respect to B.

Fact 1. The abelian inner ideals of L contained in B are just the inner ideals of the Jordan pair $\mathrm{Sub}_L B$ contained in B.

The *subquotient functor* Sub : $(\mathcal{L}, \square) \to \mathcal{JP}$ is defined by
(i) $\mathrm{Sub}(L, B) := \mathrm{Sub}_L(B)$ for any $(L, B) \in (\mathcal{L}, \square)$, and
(ii) $\mathrm{Sub}(f, f_B) := (f_B, \bar{f})$, for any $(f, f_B) \in \mathrm{Morph}((L; B), (M, C))$,
where $\bar{f}(\bar{x}) = \overline{f(x)}$ for all $x \in L$.

- It is routine to show that

$$(f_B, \bar{f}) : (B, L/\operatorname{Ker}_L B) \to (C, M/\operatorname{Ker}_M C)$$

is a homomorphism of JPs.

Fact 2. It follows from functoriality that if B and C, abelian inner ideals of L and M respectively, are conjugate under an isomorphism $f : L \to M$, then the respective subquotients $\mathrm{Sub}_L\, B$ and $\mathrm{Sub}_M\, C$ are isomorphic Jordan pairs. But the converse is not true.

$\mathbb{F}$: field of characteristic not 2.
X: Left vector space over $\mathbb{F}$.
$\langle \, , \rangle : X \times X \to \mathbb{F}$: nondegenerate symmetric bilinear form.
$\mathcal{L}(X) = \{a \in \mathrm{End}_{\mathbb{F}}(X)$ having an adjoint $a^* \in End_{\mathbb{F}}(X)\}$:

$$\langle xa, y \rangle = \langle x, ya^* \rangle, \;\; x, y \in X.$$

- $\mathcal{L}(X)$ is a primitive ring with involution $*$ and socle the ideal

$$\mathcal{F}(X) = \{a \in \mathcal{L}(X) : \dim Xa < \infty\}.$$

- As rings, $\mathcal{F}(X) \cong X \otimes_{\mathbb{F}} X$ under the product defined by

$$(w_1 \otimes v_1)(w_2 \otimes v_2) = w_1 \otimes \langle v_1, w_2 \rangle v_2,$$

where $x(w \otimes v) := \langle x, w \rangle v$, for all $x \in X$, with $(w \otimes v)^* = v \otimes w$.

- $\mathcal{L}(X) = \mathrm{End}_{\mathbb{F}}(X)$ if X is finite dimensional.

- $\mathfrak{o}(X) = \mathrm{Skew}(\mathcal{L}(\mathrm{X}), *) = \{\mathrm{a} \in \mathcal{L}(\mathrm{X}), \mathrm{a}^* = -\mathrm{a}\} \leq \mathcal{L}(\mathrm{X})^-$.
- $\mathfrak{fo}(X) = \mathrm{Skew}(\mathcal{F}(\mathrm{X}), *) \trianglelefteq \mathfrak{o}(\mathrm{X})$.
- For $x, y \in X$, set $[x, y] := x \otimes y - y \otimes x$. We have:

$$\mathfrak{fo}(X) = [X, X], \text{ linear span of all } [x, y].$$

It is worthy to notice the two meanings of the bracket notation, illustrated by the identity $(*)$:

$$\begin{aligned}[][[u_1, u_2], [v_1, v_2]] = \langle u_2, v_1\rangle[u_1, v_2] + \langle u_1, v_2\rangle[u_2, v_1] \\ - \langle u_2, v_2\rangle[u_1, v_1] - \langle u_1, v_1\rangle[u_2, v_2].\end{aligned}$$

for all $u_1, u_2, v_1, v_2 \in X$.

The lattice of the abelian inner ideals of $\mathfrak{fo}(X)$ is determined by the totally isotropic subspaces of X.

• Let $S \leq X$ be totally isotropic, $\dim S > 1$. Then $[S, S]$ is an isotropic inner ideal of $\mathfrak{o}(X)$ contained in $\mathfrak{fo}(X)$. Moreover, $[S, S]$ is a point space if and only if $\dim S = 2$ or $\dim S = 3$.

• Let $S \leq X$ be totally isotropic, $\dim S > 1$, and $s \neq 0 \in S$. Then $[s, S] = \{[s, t] : t \in S\} \subset \mathfrak{fo}(X)$ is a point space of $\mathfrak{o}(X)$ and an isotropic inner ideal.

• Let H be a hyperbolic plane of X and $0 \neq x \in H$ be isotropic. Then $[x, H^{\perp}] \subset \mathfrak{fo}(X)$ is a (maximal) abelian inner ideal of $\mathfrak{o}(X)$ called a *Clifford inner ideal*. Moreover, any proper inner ideal of $[x, H^{\perp}]$ is a point space: $P = [x, S]$, $S \leq H^{\perp}$ totally isotropic.

Let X be n-dimensional ($n = 2m$ or $2m + 1$) and p the Witt index of $\langle\,,\rangle$, $0 \leq p \leq m$. Then $X = H_1 \perp \ldots \perp H_p \perp W$, where the H_i are hyperbolic planes with hyperbolic basis (x_i, y_i) and W is anisotropic.

• Any abelian inner ideal B of $\mathfrak{o}_{p,n}(\mathbb{F})$ is conjugate to one of the following:

(i) $I_k := [S_k, S_k]$, $2 \leq k \leq p$, where $S_k = \mathbb{F}x_1 + \cdots + \mathbb{F}x_k$ and $\dim I_k = \frac{k(k-1)}{2}$.

(ii) $P_i := [x_1, S_{i+1}]$, $1 \leq i < p$, with $\dim P_i = i$.

(iii) $C_{n,p} := [x_1, H_1^{\perp}]$, with $\dim C_{n,p} = n - 2$.

• In (i) and (ii), $b^2 = 0$ for all $b \in B$ (they are isotropic inner ideals); in (iii), $b^3 = 0$ for all $b \in B$, with $c^2 \neq 0$ for some $c \in B$: $c = [x_1, z]$, where $z \in H_1^{\perp}$ is anisotropic.

Up to conjugation, this is the lattice of the abelian inner ideals of $\mathfrak{o}_{n,p}$ for each value of $n > 4$ and $p > 0$ (for $p = 0$ the only abelian inner ideal is the trivial):

(i) For $p = 1$ $\;C_{n,1}$ $\;(H_1^{\perp}$ is anisotropic).

(ii) For $p = 2$, $\;P_1 \rightarrow C_{n,2}$ $\;$(Witt index of $H_1^{\perp}$ equals 1).

(iii) For $p = 3$,

$$\begin{tikzcd} & & I_3 \\ P_1 \arrow[r] & P_2 \arrow[ur] \arrow[dr] & \\ & & C_{n,3} \end{tikzcd}$$

(iv) For $p = 4$ and $n = 8$, i.e. D_4.

$$P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow C_{8,4}.$$

As will be seen, the abelian inner ideals I_4 and $C_{8,4}$ are conjugate under a diagram automorphism which takes I_3 to P_3.

(v) For $p = 4$ and $n > 8$,

$$\begin{tikzcd} & & I_3 \arrow[r] & I_4 \\ P_1 \arrow[r] & P_2 \arrow[ur] \arrow[dr] & & \\ & & P_3 \arrow[uur] \arrow[r] & C_{n,4} \end{tikzcd}$$

As proved in [2], there are 6 abelian inner ideals for $p = 4$ and $n > 8$. Therefore, unlike the case of D_4, the point spaces P_3 and I_3 are not conjugate in (v).

(vi) For $p \geq 5$,

$$
\begin{tikzcd}
 & & I_3 \arrow[r] & I_4 \arrow[r] & I_5 \arrow[r] & \cdots \arrow[r] & I_p \\
P_1 \arrow[r] & P_2 \arrow[ur] \arrow[dr] & & & & & \\
 & & P_3 \arrow[r] \arrow[uur] & P_4 \arrow[r] \arrow[uur] & \cdots \arrow[r] & P_{p-1} \arrow[r] \arrow[uur] & C_{n,p}
\end{tikzcd}
$$

Again in this case, the point spaces I_3 and P_3 are not conjugate. While I_3 is a maximal point space, P_3 is contained in the point space P_4. However, as we will prove now, I_3 and P_3 are conjugate in D_4.

Lattice of the abelian inner ideals of D_4 up to conjugation (1)

Set $\mathrm{D}_4 = \mathfrak{o}(\mathrm{X})$, where $X = H_1 \perp H_2 \perp H_3 \perp H_4$ and each H_i a hyperbolic plane with hyperbolic basis (x_i, y_i). Choose the Cartan subalgebra $\mathfrak{H} = \sum_{i=1}^{4} \mathbb{F}[x_i, y_i]$. Then the elements

$$e_1 = [x_1, y_2],\ e_2 = [x_2, y_3],\ e_3 = [x_3, y_4],\ e_4 = [x_3, x_4]$$

are eigenvectors of $\mathrm{ad}_{\mathfrak{H}}$ with respective roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$:

$$\mathrm{ad}_h e_i = \alpha_i(h) e_i, \text{ for all } h \in \mathfrak{H} \text{ and } 1 \le i \le 4.$$

In fact, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis of the root system of D_4 and the map $\alpha_1 \to \alpha_3 \to \alpha_4 \to \alpha_1$ fixing the second node α_2 is an automorphism of the Dynkin diagram of D_4

$$\begin{tikzcd}
 & & \alpha_4 \\
\alpha_1 \arrow[r] & \alpha_2 \arrow[ur] \arrow[dr] & \\
 & & \alpha_3
\end{tikzcd}$$

Denote by φ the corresponding automorphism of D_4:
$[x_1, y_2] = e_1 \mapsto e_3 = [x_3, y_4]$, $[x_2, y_3] = e_2 \mapsto e_2 = [x_2, y_3]$,
$[x_3, y_4] = e_3 \mapsto e_4 = [x_3, x_4]$, (i) $[x_3, x_4] = e_4 \mapsto e_1 = [x_1, y_2]$.
We have

(1) $[x_1, y_3] = [e_1, e_2] \mapsto [e_3, e_2] = [[x_3, y_4], [x_2, y_3]] = [y_4, x_2]$,

(2) $[x_2, y_4] = [e_2, e_3] \mapsto [e_2, e_4] = [[x_2, y_3], [x_3, x_4]] = [x_2, x_4]$.

- φ moves I_4 to $C_{4,8}$ and I_3 to P_3.

(ii) $[x_2, x_4] = [e_2, e_4] \mapsto [e_2, e_1] = [[x_2, y_3], [x_1, y_2]] = -[x_1, y_3]$,

(iii) $[x_1, x_4] = [[x_1, y_3], [x_3, x_4]] \mapsto [[y_4, x_2], [x_1, y_2]] = [x_1, y_4]$,

(iv) $[x_2, x_3] = [[x_3, x_4], [x_2, y_4]] \mapsto [[x_1, y_2], [x_2, x_4]] = [x_1, x_4]$,

(v) $[x_1, x_2] = [[x_2, y_4], [x_1, x_4]] \mapsto [[x_2, x_4], [x_1, y_4]] = [x_1, x_2]$,

(vi) $[x_1, x_3] = [[x_3, y_4], [x_1, x_4]] \mapsto [[x_3, x_4], [x_1, y_4]] = [x_1, x_3]$,

where we have used the identities $(*)$, (1) and (2).

It is noted in the classification of the simple finite dimensional Jordan pairs over an algebraically closed field of characteristic zero $\mathbb{F}$ [4, 17.11(5)]:

The Jordan pair $\mathcal{K}_4 := (K_4, K_4)$ *of skew-symmetric matrices of size 4 with coefficients in* $\mathbb{F}$ *is isomorphic to the Jordan pair* $\mathcal{Q}_6$ *of a* 6*-dimensional* $\mathbb{F}$*-vector space with a nondegenerate symmetric bilinear form.*

This Jordan pair isomorphism can be derived from the fact that the automorphism φ of D_4 sending I_4 to $C_{8,4}$ induces (by functoriality) the Jordan pair isomorphism

$$(\varphi_{I_4}, \bar{\varphi}) : \ \mathrm{Sub}_{\mathrm{D}_4} I_4 \to \mathrm{Sub}_{\mathrm{D}_4} C_{8,4},$$

with $\mathrm{Sub}_{\mathrm{D}_4} I_4 \cong \mathcal{K}_4$ and $\mathrm{Sub}_{\mathrm{D}_4} C_{8,4} \cong \mathcal{Q}_6$, as proved in [1].

Jordan classification of abelian inner ideals (1)

Let (L, B) and (M, C) be objects of the category $(\mathcal{L}, \square)$ over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2, 3$. The abelian inner ideals B and C are called *Jordan isomorphic* if its respective subquotients are isomorphic Jordan pairs.

An abelian inner ideal B of L is called *Clifford* (resp. *isotropic*) if $\mathrm{Sub}_L\, B$ is a Clifford Jordan pair (resp. a Jordan pair of skew matrices).

Using this terminology, we have that E_6 and E_7 have Clifford inner ideals (not maximal abelian) of dimensions 8 and 10 respectively; while E_8 and F_4 contain Clifford inner ideals (maximal abelian) of dimensions 14 and 7 respectively; G_2 just contains two points spaces of dimensions 1 and 2, [3].

Jordan classification of abelian inner ideals (2)

Abelian inner ideals don't like to live in an exceptional Lie algebras, they prefer to move to a special one.

Fact 3. An abelian inner ideal B living in a large house L moves to a flat T

$$B \to V = \mathrm{Sub}_L\, B \to T = TKK(V) = T_{-1} \oplus T_0 \oplus T_1,$$

which fits better with its real necessities. And in its new home T, the abelian inner ideal $B = T_1$ will no longer be a single: it will find a partner, the abelian inner ideal T_{-1}.

Take in particular $L = E_6$ and B its 8-dimensional Clifford inner. The new home T of B is the orthogonal Lie algebra D_5.

Notice that while $\dim E_6 = 78$, the dimension of D_5 is 45.

Referencias

- G. Benkart and A. Fernández López, *The Lie inner ideal structure of associative rings revisited*, Comm. Algebra **37** (2009), 3833-3850.
- C. Draper and J. Meulewaeter, *Inner ideals of real simple Lie algebras*, Bull. Malays. Math. Soc. **45** (2022), 2313-2345.
- A. Fernández López, *Jordan Structures in Lie Algebras*, Mathematical Surveys and Monographs, vol **240**, AMS, 2019.
- Ottmar Loos, *Jordan pairs*, Lecture Notes in Mathematics, vol **460**, Springer-Verlag, Berlin, New York, 1975.