Some recent results on evolution algebras

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Evolution algebras

Definition

$B = \{e_1, \dots, e_n\}$ such that

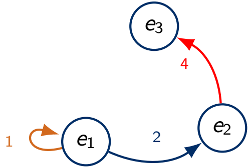
$$e_i e_j = 0 \quad (i \neq j), \qquad e_i^2 = \sum_{k=1}^{n} w_{ik} e_k .$$

The basis B is called **natural basis** *and $W_B = (w_{ik})$ is the structure matrix with respect to B.*

1. Weighted directed graph: vertex i, edge $i \to k$ of weight w_{ik}.
2. For example: $\{e_1, e_2, e_3\}$

$$e_1^2 = e_1 + 2e_2, \qquad e_2^2 = 4e_3.$$

3. (Elduque, Labra) Nilpotent if and only if no oriented cycles.



Question

How big is the class of evolution algebras compared to commutative algebras?

The class of evolution algebras is not a variety!

1. **Homomorphic images:** if $A \in \mathcal{C}$ and $\varphi : A \to B$ is a surjective homomorphism, then $B \in \mathcal{C}$.
2. **Direct products:** if $\{A_i\}_{i \in I} \subseteq \mathcal{C}$, then $\prod_{i \in I} A_i \in \mathcal{C}$.
3. **Subalgebras:** if $A \in \mathcal{C}$ and $B \leq A$, then $B \in \mathcal{C}$.

1. (Ladra-Pérez Rodríguez) Regular ($E^2 = E$) finite-dimensional evolution algebras are closed under subalgebras.
2. Even if we can prove it for any dimension the direct product condition still fails.

Remark: A stronger statement

The previous obstruction says that evolution algebras cannot be *defined* by identities.

But the situation is worse.

Theorem

Let p be a non-zero multilinear commutative polynomial of degree $d \geq 2$. Then there exists an evolution algebra E of dimension $d - 1$ that does not satisfy the identity.

Corollary

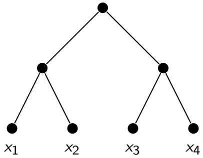
The class of all evolution algebras satisfies no non-trivial polynomial identity beyond commutativity.

Proof idea: monomials as trees

We work in the free commutative non-associative algebra

$$\mathbb{F}\{x_1, \ldots, x_d\}.$$

A multilinear monomial of degree d can be represented by a rooted binary tree whose leaves are labelled by the variables. For example $(x_1x_2)(x_3x_4)$



Thus a polynomial identity is a linear combination of such trees:

$$p = \sum_T \alpha_T T.$$

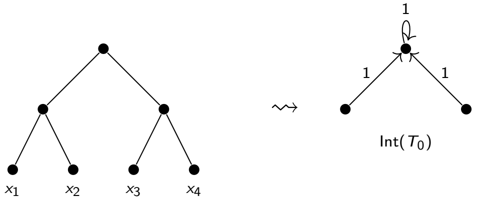
Strategy

Choose one tree T_0 with $\alpha_{T_0} \neq 0$, and construct an evolution algebra which separates T_0 from all the other trees.

From a tree to an evolution algebra

Start with the chosen labelled tree T_0.

1. Cut off the leaves.
2. Keep the interior tree $\text{Int}(T_0)$.
3. Orient every edge towards the root and add one loop at the root.
4. Put weight 1 on every edge.



This weighted directed graph defines an evolution algebra S_{T_0}.

Why this algebra separates the chosen monomial

The evaluation is chosen according to the leaves of T_0: each variable is sent to the vertex just above its corresponding leaf.

Separation property

For the chosen tree T_0,

$$T_0\Big|_{\varphi_{T_0}} \neq 0.$$

For every other labelled tree T,

$$T\Big|_{\varphi_{T_0}} = 0 \qquad \text{unless } T \cong T_0.$$

Therefore, if

$$p = \sum_T \alpha_T T$$

and $\alpha_{T_0} \neq 0$, then the evaluation on S_{T_0} leaves only the T_0-term:

$$p\Big|_{\varphi_{T_0}} \neq 0.$$

Hence $p = 0$ cannot be an identity.

So how big is the class?

We now fix the dimension. Let $\mathcal{C}_n$ be the affine variety of n-dimensional commutative algebras, written through structure constants. Thus, after fixing a basis of $V := \mathbb{C}^n$, each point of $\mathcal{C}_n$ is a product

$$\mu : V \times V \longrightarrow V, \qquad \mu(x, y) = \mu(y, x).$$

Inside $\mathcal{C}_n$ we consider

$$\mathrm{Ev}_n = \{\, \mu \in \mathcal{C}_n : (V, \mu) \text{ admits a natural basis} \,\}.$$

Theorem. Density

Over $\mathbb{C}$, one has

$$\overline{\mathrm{Ev}_2}^{\,\mathrm{Zar}} = \mathcal{C}_2,$$

whereas, for $n > 2$, we have $\overline{\mathrm{Ev}_n}^{\,\mathrm{Zar}} \subsetneq \mathcal{C}_n$.

Identities in Ev_n

Now fix the dimension n, and consider the class Ev_n of n-dimensional evolution algebras.

Low degrees

Ev_n satisfies no polynomial identity of degree $k \leq n+1$.

At degree $n+2$, an identity appears: the non-associative standard identity

$$\mathrm{st}_{n+2}(x_1, \ldots, x_{n+2}) = 0.$$

Here

$$\mathrm{st}_k(x_1, \ldots, x_k) = \sum_{\sigma \in S_{k-1}} \mathrm{sign}(\sigma)\Big((((x_k x_{\sigma(k-1)})x_{\sigma(k-2)}) \cdots)x_{\sigma(1)}\Big).$$

Theorem

The minimal degree of a polynomial identity for Ev_n is $n+2$.

Remark

The first identity appears as late as possible: it is not specific to evolution algebras, but is forced by the dimension, since every n-dimensional non-associative algebra satisfies $\mathrm{st}_k = 0$ for $k \geq n+2$.

Embedding commutative algebras into evolution algebras as ideals

Theorem

Let A be a finite-dimensional commutative k-algebra, with $\dim A = n$. Then there exists an evolution k-algebra X of dimension

$$\frac{n(n+1)}{2}$$

and an algebra monomorphism $\psi : A \hookrightarrow X$ such that $X^2 \subseteq \psi(A)$.

Definition

An *evolution envelope* of a commutative algebra A is a pair (E, Φ) where E is an evolution algebra, $\Phi : A \hookrightarrow E$ is an algebra monomorphism, and $E^2 \subseteq \Phi(A)$.

Proposition

If A embeds into an evolution algebra E, then, on the same underlying vector space, one can modify the product to obtain an evolution envelope of A.

Idea: Choose a linear projection $\pi : E \to \Phi(A)$ and define a new product by projecting the old one.

Elementary consequences of envelopes

Proposition

Every finite-dimensional commutative algebra can be represented inside an evolution algebra, and hence by a graph! (not unique or canonical in any way tho)

Let (E, Φ) be an evolution envelope of A.

Proposition

For the derived series $A^{(1)} = A$, $A^{(k+1)} = A^{(k)}A^{(k)}$, one has for $k \geq 2$

$$\Phi(A^{(k)}) \subseteq E^{(k)} \subseteq \Phi(A^{(k-1)}).$$

Moreover:

1. *if A is perfect, then $A \cong E^2$;*
2. *A is solvable if and only if E is solvable;*
3. *Φ induces a bijection between idempotents of A and idempotents of E.*

Application: solvability problems

After constructing evolution envelopes, a natural question was:

Can classical solvability problems be translated into the language of evolution algebras?

Using evolution envelopes, one easily obtain the following reformulations.

Albert problem

The following are equivalent:

1. Every finite-dimensional commutative power-associative nil algebra is solvable.
2. Every finite-dimensional evolution algebra E such that E^2 is nil and power-associative is solvable.

Albert–Engel problem

The following are equivalent:

1. Every finite-dimensional commutative Engel algebra is solvable.
2. Every finite-dimensional evolution algebra E such that E^2 is Engel is solvable.

Around this point, García-Martínez and Pérez-Rodríguez proposed a stronger solvability conjecture for evolution algebras:

$$E \text{ solvable} \quad \Longleftrightarrow \quad E \text{ has no non-zero idempotents.}$$

If true, it would imply both Albert-type problems (and some other big ones).

Application: solvability problems

Using evolution envelopes, one can transfer counterexamples from commutative algebras to evolution algebras.

Example

Consider the complex commutative algebra A with basis $\{b_1, b_2\}$ and multiplication

$$b_1^2 = 0, \qquad b_1 b_2 = b_1, \qquad b_2^2 = b_1 + 2b_2.$$

Then A is not solvable and has no non-zero idempotents.

A minimal evolution envelope (E, φ) of A has natural basis $\{e_1, e_2, f\}$, with

$$e_1^2 = -e_1 - f, \qquad e_2^2 = 2e_2 + 2f, \qquad f^2 = e_1 + f,$$

and embedding

$$\varphi(b_1) = e_1 + f, \qquad \varphi(b_2) = e_2 + f.$$

Conclusion

The evolution algebra E is not solvable and has no non-zero idempotents. Hence the conjecture is false.

From commutative algebras to quadratic maps

Assume $\mathrm{char}(k) \neq 2$. A finite-dimensional commutative algebra A can be encoded by the quadratic map

$$H_A : A \longrightarrow A, \qquad H_A(x) = x^2.$$

Conversely, the product is recovered from H_A by polarization:

$$xy = \frac{1}{2}\big(H_A(x+y) - H_A(x) - H_A(y)\big).$$

After fixing a basis, H_A becomes a tuple of homogeneous quadratic forms.

Example

Let $A = k[t]/(t^4)$, and write $x = x_0 + x_1 t + x_2 t^2 + x_3 t^3$. Then

$$H_A(x) = x^2 = \big(x_0^2,\, 2x_0x_1,\, 2x_0x_2 + x_1^2,\, 2x_0x_3 + 2x_1x_2\big).$$

If E is an evolution algebra with structure matrix W, then in a natural basis

$$H_E(y_1, \ldots, y_m) = W^T \begin{pmatrix} y_1^2 \\ \vdots \\ y_m^2 \end{pmatrix}.$$

So evolution algebras correspond to quadratic maps whose coordinates are linear combinations of squares!

Simultaneous Waring decompositions

The previous observation suggests the right question:

Can the quadratic map H_A be written using only squares of linear forms?

Definition

A *simultaneous Waring decomposition* of H_A is an expression

$$H_A(x) = \sum_{j=1}^{m} \ell_j(x)^2 v_j, \qquad \ell_j \in A^*, \quad v_j \in A.$$

The minimal possible length m is the Waring rank $wr(H_A)$.

Equivalently, this decomposition can be written as

$$H_A = D \circ L,$$

where

$$L : A \longrightarrow k^m, \qquad L(x) = \big(\ell_1(x), \ldots, \ell_m(x)\big),$$

and

$$D : k^m \longrightarrow A, \qquad D(y_1, \ldots, y_m) = \sum_{j=1}^{m} y_j^2 v_j.$$

Minimal envelopes via Waring decompositions

The decomposition

$$H_A = D \circ L$$

has two pieces:

- $L : A \to k^m$ is the candidate embedding;
- $D : k^m \to A$ records the squares of the natural basis vectors.

From D and L, we put an evolution algebra structure on k^m by taking the quadratic map

$$H_E = L \circ D : k^m \longrightarrow k^m.$$

Then the standard basis of k^m is natural, and

$$L : A \hookrightarrow E$$

is an evolution envelope exactly when L is injective.

Definition

Let $sr(H_A)$ be the minimal length m of a simultaneous Waring decomposition $H_A = D \circ L$ with L injective.

Theorem

The minimal dimension of an evolution envelope of A is $sr(H_A)$.

Minimal envelopes via Waring decompositions

Relation with the Waring rank

If $wr(H_A)$ denotes the usual minimal Waring length, then

$$sr(H_A) = wr(H_A) + \dim \mathrm{Ann}(A).$$

Theorem

The minimal dimension of an evolution envelope for $A = k[t]/(t^n)$ is $2n-1$.

Thank you.

Congratulations Manolo on your 70th birthday.