Quantum immanants for the queer Lie superalgebra

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(joint with Alexander Molev and Vera Serganova,

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Sergeev superalgebra

The *Sergeev superalgebra* $\mathcal{S}_n$ is the graded tensor product of two superalgebras

$$\mathcal{S}_n = \mathbb{C}\mathfrak{S}_n^- \otimes \mathcal{C}l_n,$$

where $\mathbb{C}\mathfrak{S}_n^-$ is generated by odd elements $t_1, \ldots, t_{n-1}$

$$t_a^2 = 1, \qquad t_a t_{a+1} t_a = t_{a+1} t_a t_{a+1}, \qquad t_a t_b = -t_b t_a, \qquad |a-b| > 1,$$

while $\mathcal{C}l_n$ is generated by odd elements $c_1, \ldots, c_n$

$$c_a^2 = -1, \qquad c_a c_b = -c_b c_a, \qquad a \neq b.$$

Jucys–Murphy elements

Introduce analogs of the transpositions in the Sergeev superalgebra

$$t_{ab} = (-1)^{b-a-1}\, t_{b-1} \dots t_{a+1} t_a t_{a+1} \dots t_{b-1}, \quad a < b, \quad t_{ba} = -t_{ab}.$$

The odd Jucys–Murphy elements are

$$m_1 = 0, \qquad m_a = t_{1a} + \dots + t_{a-1,a}, \qquad a = 2, \dots, n.$$

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and the even Jucys–Murphy elements are

$$x_1 = 0, \quad x_a = \sqrt{2}\,(t_{1a} + \dots + t_{a-1,a})\, c_a, \quad a = 2, \dots, n.$$

Simple modules over $\mathcal{S}_n$ and $\mathbb{C}\mathfrak{S}_n^-$ are parameterized by strict partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of n with $\lambda_1 > \cdots > \lambda_\ell > 0$ and $\lambda_1 + \cdots + \lambda_\ell = n$.

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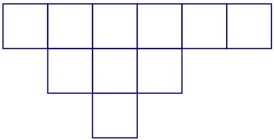
Depict λ by the shifted Young diagram: e.g. for $\lambda = (6, 3, 1)$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The dimensions of the simple modules over $\mathcal{S}_n$ and $\mathbb{C}\mathfrak{S}_n^-$:

$$\dim U^{\lambda} = 2^{n-\lfloor \frac{\ell(\lambda)}{2} \rfloor} g_{\lambda} \qquad \text{and} \qquad \dim V^{\lambda} = 2^{\lceil \frac{n-\ell(\lambda)}{2} \rceil} g_{\lambda},$$

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where g_λ is the number of standard λ-tableaux, found by the Schur formula [1911] :

$$g_\lambda = \frac{n!}{\lambda_1! \ldots \lambda_\ell!} \prod_{1\leqslant i<j\leqslant\ell} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

We need to extend the set of tableaux by allowing any non-diagonal entry to occur with a bar on it [Sagan 1987]:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & \overline{4} & \overline{5} & 8 & \overline{10} \\ \hline \multicolumn{1}{c|}{} & 3 & \overline{6} & 9 & \multicolumn{2}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 7 & \multicolumn{3}{c}{} \\ \cline{3-3} \end{array} .$$

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The signed content of any barred or unbarred entry a of $\mathcal{U}$ is

$$\kappa_{\mathsf{a}}(\mathcal{U}) = \begin{cases} \sqrt{\sigma_a(\sigma_a+1)} & \text{if } \mathsf{a} = a \text{ is unbarred,} \\ -\sqrt{\sigma_a(\sigma_a+1)} & \text{if } \mathsf{a} = \bar{a} \text{ is barred,} \end{cases}$$

where $\sigma_a = j - i$ is the content of the box (i,j) of λ with a or $\bar{a}$.

Idempotents in Sergeev superalgebra

For any standard barred tableau $\mathcal{U}$ set $e_{\boxed{1}} = 1$ and

$$e_{\mathcal{U}} = e_{\mathcal{V}} \, \frac{(x_n - b_1) \dots (x_n - b_p)}{(\kappa - b_1) \dots (\kappa - b_p)},$$

where $\mathcal{V}$ is obtained from $\mathcal{U}$ by removing the box α occupied by n (or $\bar{n}$). The shape of $\mathcal{V}$ denote by μ, then $b_1, \dots, b_p$ are the signed contents in all addable boxes of μ, except n ($\bar{n}$).

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Theorem (K, Molev, Serganova, 2025)

$$e_{\mathcal{U}} e_{\mathcal{V}} = \delta_{\mathcal{U}\mathcal{V}} e_{\mathcal{V}}, \qquad 1 = \sum_{\lambda \Vdash n} \sum_{\mathrm{sh}(\mathcal{U}) = \lambda} e_{\mathcal{U}}.$$

Moreover, $x_{\mathsf{a}}\, e_{\mathcal{U}} = e_{\mathcal{U}}\, x_{\mathsf{a}} = \kappa_{\mathsf{a}}(\mathcal{U})\, e_{\mathcal{U}}$.

Schur–Weyl duality

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$$\operatorname{tr} \mathcal{E}_{\mathcal{U}} Y_1 \ldots Y_n = s_\lambda(y_1, \ldots, y_N),$$

where $Y = \operatorname{diag}(y_1, \ldots, y_N)$ and $Y_a = 1^{\otimes(a-1)} \otimes Y \otimes 1^{\otimes(n-a)}$.

Sergeev duality

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Consider the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{N|N}$ with the canonical basis $e_{-N}, \ldots, e_{-1}, e_1, \ldots, e_N$ with $p(e_i) = \bar{\imath} \mod 2$, where

$$\bar{\imath} = 1 \quad \text{for} \quad i < 0 \qquad \text{and} \qquad \bar{\imath} = 0 \quad \text{for} \quad i > 0.$$

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The $\mathcal{S}_n=\mathbb{C}\mathfrak{S}_n \ltimes \mathcal{C}l_n$ acts on the space

$$\underbrace{\mathbb{C}^{N|N}\otimes\dots\otimes\mathbb{C}^{N|N}}_{n}$$

by $\qquad c_a \mapsto J_a \qquad$ and $\qquad (a,b)\mapsto P_{ab}, \quad$ with

$$J=\sum_{i=-N}^{N} E_{i,-i}(-1)^{\bar{\imath}} \in \operatorname{End}\mathbb{C}^{N|N}.$$

The superalgebra $Q_N \subset \operatorname{End}\mathbb{C}^{N|N}$ has the basis

$$e_{kl} = E_{kl} + E_{-k,-l}, \qquad f_{kl} = E_{k,-l} + E_{-k,l}, \qquad 1 \leqslant k, l \leqslant N,$$

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and $Q_\lambda(y_1, \dots, y_N)$ is the Schur Q-polynomial.

Factorial Schur Q-polynomials

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Use the alphabet $1' < 1 < 2' < 2 < \cdots < N' < N$ and set

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the labels increase weakly in rows and columns.

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The factorial Schur Q-polynomial is given by

$$Q_\lambda^+(y) = \sum_{\mathrm{sh}(\mathcal{T})=\lambda} \prod_{\alpha\in\lambda} \big(y_{\mathcal{T}(\alpha)} + \mathrm{sgn}\,(\mathcal{T}(\alpha))\sigma(\alpha)\big),$$

where $\sigma(\alpha) = j - i$ is the content of $\alpha = (i, j)$ [Ivanov, 2005].

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The top degree component of $Q_\lambda^+(y)$ is $Q_\lambda(y)$

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Vanishing property:

$$Q_\lambda^+(-\mu) = 0$$

for all strict μ with $\ell(\mu) \leqslant N$ and $|\mu| < |\lambda|$.

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Characterization property [Ivanov, 2005]:

If the top degree component of a polynomial $P(y) \in \Gamma_N$ coincides with $Q_\lambda(y)$ for some λ with $\ell(\lambda) \leqslant N$, and $P(-\mu) = 0$ for all strict μ with $\ell(\mu) \leqslant N$ and $|\mu| < |\lambda|$, then $P(y) = Q^+_\lambda(y)$.

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Lie superalgebras $\mathfrak{q}_N$ and $\mathfrak{q}_M$ act on $\mathcal{P}$; for $\mathfrak{q}_N$ we have

$$e_{kl} \mapsto \sum_{a=-M}^{M} x_{ak} \partial_{la},$$

$$f_{kl} \mapsto \sum_{a=-M}^{M} x_{ak} \partial_{l,-a},$$

with $k, l = 1, \dots, N$.

We have a decomposition for polynomials of degree n:

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In particular, the highest weight $\mathfrak{q}_N$-modules $L(\lambda)$ can be realized as submodules of $\mathcal{P}^n$ if $\ell(\lambda) \leqslant M$ and $\ell(\lambda) \leqslant N$.

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This defines another action of $\mathfrak{q}_N$ in $\mathcal{P}$,

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We thus get a homomorphism $\mathrm{U}(\mathfrak{q}_N) \to \mathcal{PD}$.

Combine the variables and derivations into matrices X and D:

$$X = \sum_{a=1}^{M} \sum_{k=1}^{N} \left(e_{ka} \otimes x_{ak} - i f_{ka} \otimes x_{-a,k} \right) \in \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^{N|N}) \otimes \mathcal{PD}$$

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Introduce the odd element

$$G = \sum_{k,l=1}^{N} \left(e_{kl} \otimes f_{lk} - f_{kl} \otimes e_{lk} \right) \in Q_N \otimes \mathrm{U}(\mathfrak{q}_N).$$

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This is the image of the odd Jucys–Murphy element.

Theorem. Under the action $G \mapsto XD$, we have

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Inspired by the universal Capelli identity for $\mathfrak{gl}_N$ [Zaitsev, 2025].

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Corollary. Under the action $F \mapsto XD$, we have

$$\big(F_1 + \kappa_1(\mathcal{U})\big) \ldots \big(F_n + \kappa_{\mathsf{n}}(\mathcal{U})\big)\, \mathcal{E}_{\mathcal{U}} \mapsto X_1 \ldots X_n D_1 \ldots D_n\, \mathcal{E}_{\mathcal{U}},$$

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$$\kappa_{\mathsf{a}}(\mathcal{U}) = \begin{cases} \sqrt{\sigma_a(\sigma_a + 1)} & \text{if } \ \mathsf{a} = a \ \text{ is unbarred,} \\ -\sqrt{\sigma_a(\sigma_a + 1)} & \text{if } \ \mathsf{a} = \bar{a} \ \text{ is barred,} \end{cases}$$

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Proof. $\mathcal{X}^{(n)}\mathcal{E}_{\mathcal{U}} = \kappa_{\mathsf{n}}(\mathcal{U})\,\mathcal{E}_{\mathcal{U}}$.

Corollary. Under the action $F \mapsto XD$, we have

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Proof. $\mathcal{X}^{(n)}\mathcal{E}_\mathcal{U} = \kappa_\mathsf{n}(\mathcal{U})\,\mathcal{E}_\mathcal{U}$. Cf. [Okounkov, 1996] for $\mathfrak{gl}_N$.

Quantum immanants

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Moreover, it depends only on λ and does not depend on $\mathcal{U}$.

Recall the Harish-Chandra isomorphism [Sergeev, 1999]:

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On any highest vector $\xi \in L(y)$ we have

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The scalar is a supersymmetric polynomial denoted by $\chi(\mathbb{S}_\lambda)$.

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Taking into account the twisting of the action of $\mathfrak{q}_N$ coming from the queer Howe duality, the element $\mathbb{S}_\lambda$ acts as zero in all $L(-\widetilde{\mu})$ with $|\mu| < |\lambda| = n$, where $\widetilde{\mu} = (\mu_N, \ldots, \mu_1)$.

Feliz Aniversário, Manolo!