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Poisson derivations of Poisson nilpotent algebras

ManoloFest – Categorical, homological and computational methods in Algebra

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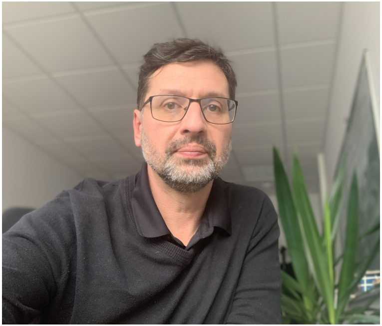
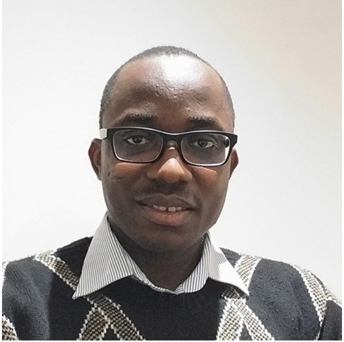
meet the coauthors



S. Launois (Caen)



I. Oppong (Greenwich)

Moment 0: motivation and background

Poisson algebras

Definition. *Siméon Denis Poisson (1809)*
A Poisson algebra over $\Bbbk$ is a unital associative commutative $\Bbbk$-algebra $(A, \cdot)$ endowed with a Lie bracket $\{\ ,\ \} : A \times A \to A$ (the Poisson bracket) such that $\{a, -\}$ is a derivation of $(A, \cdot)$, for every $a \in A$.

Most notions for Lie and associative algebras are extended to Poisson algebras by viewing them both in $(A, \cdot)$ and in $(A, \{\ ,\ \})$:

- A Poisson ideal of A is an ideal J such that $\{A, J\} \subseteq J$.
- The Poisson center of A is $Z_P(A) = \{z \in A \mid \{z, A\} = 0\}$.
- A Poisson morphism is a unital (associative) $\Bbbk$-algebra morphism which is also a Lie morphism.
- A Poisson derivation is a $\Bbbk$-linear map which is both an associative algebra derivation and a Lie algebra derivation.

Examples

Poisson algebras arise naturally in mathematical physics, Poisson geometry, deformation quantization, integrable systems and noncommutative algebraic geometry.

- Poisson Weyl algebra: $PA_1 = \Bbbk[x, y]$ with $\{x, y\} = 1$.
- More generally, given two sets of variables $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$, set $PA_n = \Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with

$$\{f, g\} = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\frac{\partial g}{\partial y_k} - \frac{\partial g}{\partial x_k}\frac{\partial f}{\partial y_k}.$$

- [Kostant–Kirillov Poisson bracket] The symmetric algebra $S(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ with basis $\{e_i\}$ is a Poisson algebra under

$$\{f, g\} = \sum_{i,j} \frac{\partial f}{\partial e_i}\frac{\partial g}{\partial e_j}[e_i, e_j].$$

  The coadjoint orbits in $\mathfrak{g}^*$ of the connected, simply connected Lie group G correspond to Poisson radical ideals in $S(\mathfrak{g})$.

Semiclassical limits

Let $\mathcal{A}$ be associative (and noncommutative) and $0 \neq \hbar \in Z(\mathcal{A})$ (nonunit and not a zero divisor).

If $A = \mathcal{A}/h\mathcal{A}$ is commutative, define a Poisson bracket on A by

$$\{\overline{a}, \overline{b}\} = \overline{\hbar^{-1}[a, b]} = (\hbar^{-1}[a, b])\big|_{\hbar=0}, \quad \forall a, b \in A.$$

Endowed with this bracket, the Poisson algebra A is called the semiclassical limit of $\mathcal{A}$.

In turn, $\mathcal{A}$ is called a quantization of A.

Example The universal enveloping algebra of $\mathfrak{g}$ can thus be seen as a quantization of $S(\mathfrak{g})$.

Poisson cohomology

Poisson cohomology is a powerful invariant providing insight into the symmetries and deformations of Poisson algebras. It has the usual interpretations in low degrees:

$$
\begin{array}{lcl}
\mathrm{PH}^0(A) & = & \mathrm{Z}_P(A) \\
\mathrm{PH}^1(A) & = & \mathrm{Der}_P(A)/\mathrm{Ham}(A) \\
\mathrm{PH}^2(A) & = & \text{infinitesimal deformations of } A \\
\mathrm{PH}^3(A) & & \text{contains the obstructions to formal deformations of } A
\end{array}
$$

Here:

- $\mathrm{Der}_P(A)$ is the Lie algebra of Poisson derivations of A.
- $\mathrm{Ham}(A)$ is the ideal of Hamiltonian derivations, i.e., those of the form $\mathrm{Ham}_a = \{a, \cdot\}$, for $a \in A$.

Ultimate goal: Find techniques to compute $\mathrm{PH}^\bullet$ of Poisson quotients of a certain class of algebras — Poisson nilpotent algebras.

Moment 1: Poisson nilpotent algebras (PNA)

Goal: extend the structure from A to $A[x]$, respecting the degree.

Definition. *(Poisson Ore extension $A[x;\sigma,\delta]_P$)*
As a commutative associative algebra, $A[x;\sigma,\delta]_P = A[x]$.
The Poisson structure extends A with

$$\{x,a\} = \sigma(a)x + \delta(a), \quad \text{for } a \in A.$$

The above set-up gives a Poisson algebra extending A if and only if $\sigma \in \mathrm{Der}_P(A)$ and δ is a Poisson σ-derivation:

$$\delta(\{a,b\}) = \{\delta(a),b\} + \{a,\delta(b)\} + \sigma(a)\delta(b) - \delta(a)\sigma(b), \quad \text{for all } a,b \in A.$$

Examples

1. Given $\sigma \in \mathrm{Der}_P(A)$ and $c \in A$,

$$\delta(\,\cdot\,) = \mathrm{Ham}_c(\,\cdot\,) - \sigma(\,\cdot\,)c$$

   defines a σ-derivation of A.

2. Consider $A = \Bbbk[x]$ with trivial Poisson bracket,

$$\sigma = x\frac{d}{dx} \in \mathrm{Der}_P(A) \quad \text{and} \quad \delta = \frac{d}{dx} \in \sigma - \mathrm{Der}_P(A).$$

   Then $A[y; x\frac{d}{dx}, \frac{d}{dx}]_P$ is $\Bbbk[x, y]$, with Poisson bracket

$$\{y, x\} = xy + 1.$$

The quantum version of the coordinate ring of the plane is

$$\mathcal{O}_q(\mathbb{k}^2) = \mathbb{k}\{x, y \mid yx = q^a xy\},$$

where $q \in \mathbb{k}^\times$ and $a \in \mathbb{Z}$.

The semiclassical limit as $q \to 1$ is the Poisson affine plane $\mathbb{k}[x, y]$ with

$$\{y, x\} = \frac{1}{q-1}[y, x]\Big|_{q=1} = \frac{q^a - 1}{q-1}\Big|_{q=1} xy = axy.$$

This is the Poisson Ore extension $\mathbb{k}[x][y; ax\frac{d}{dx}, 0]$.

$M_q(2)$ is a deformation of $\mathcal{O}(M_2(\Bbbk)) = \underset{\text{(coordinate projections)}}{\underline{\Bbbk[a, b, c, d]}}$ with

(new) generators a, b, c, d and relations:

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, dash, dotted] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd}$$

where

$x \longrightarrow y$ means $yx = qxy$

$a \cdots\cdots d$ means $ad - da = (q^{-1} - q)bc$

b and c commute

The semiclassical limit as $q \to 1$ is the Poisson algebra $\Bbbk[a, b, c, d]$:

$\{a, b\} = ab, \{a, c\} = ac, \{b, c\} = 0, \{b, d\} = bd, \{c, d\} = cd, \{a, d\} = 2bc.$

This is a <u>Poisson Ore extension</u> $\Bbbk[a][b; \sigma_b]_P[c; \sigma_c]_P[d; \sigma_d, \delta_d]_P$.

The semiclassical limit of the quantum nilpotent algebra $U_q(\mathfrak{so}_5^+)$ is $A = \Bbbk[X_1, X_2, X_3, X_4]$ with Poisson brackets

$$\begin{aligned}
&\{X_2, X_1\} = -2X_1X_2, && \{X_3, X_1\} = X_2^2, && \{X_3, X_2\} = -2X_2X_3,\\
&\{X_4, X_1\} = -2X_1X_4 - X_2, && \{X_4, X_2\} = -2X_3, && \{X_4, X_3\} = -2X_3X_4.
\end{aligned}$$

Once again we have a Poisson Ore extension

$$\Bbbk[X_1][X_2; \sigma_2]_P[X_3; \sigma_3, \delta_3]_P[X_4; \sigma_4, \delta_4]_P$$

with

$$\sigma_2 = -2X_1\frac{d}{dX_1}, \qquad \sigma_3 = -2X_2\frac{d}{dX_2}, \qquad \delta_3 = X_2^2\frac{d}{dX_1},$$
$$\sigma_4 = -2X_1\frac{d}{dX_1} - 2X_3\frac{d}{dX_3}, \qquad \delta_4 = -X_2\frac{d}{dX_1} - 2X_3\frac{d}{dX_2}.$$

Poisson nilpotent algebras

A PNA (*aka* Poisson Cauchon–Goodearl–Letzter extension) is an iterated Poisson Ore extension

$$R = \underbrace{\Bbbk[x_1]_P \cdots [x_{k-1}; \sigma_{k-1}, \delta_{k-1}]_P}_{R_{k-1}}[x_k; \sigma_k, \delta_k]_P \cdots [x_N; \sigma_N, \delta_N]_P$$

equipped with a rational Poisson action (by P–automorphisms) of a torus $\mathcal{H} = (\Bbbk^\times)^n$, such that:

(i) the x_i are $\mathcal{H}$-eigenvectors;

(ii) the δ_k are locally nilpotent;

(iii) $\sigma_k = h_k \cdot|_{R_{k-1}}$, for some $h_k \in \mathrm{Lie}(\mathcal{H})$ such that the h_k-eigenvalue of x_k is nonzero.

We will always take the maximal such torus, so

$$n = |\{k \in [1, N] \mid \delta_k = 0\}| = \mathrm{rk}(R),$$

called the rank of R.

A few remarks

* The action of $\mathcal{H} = (\Bbbk^\times)^n$ is tantamount to a Q-grading, with $Q = \mathbb{Z}^n$, where $\mathcal{H}$-eigenvalues correspond to homogeneous elements.

* If S is a multiplicative set in a Poisson algebra A, then the Poisson bracket extends to the localization AS^{-1} via

$$\left\{a, s^{-1}\right\} = -s^{-2}\left\{a, b\right\}.$$

* An element $u \in A$ is Poisson normal if $\{u, A\} \subseteq uA$, so that uA is a Poisson ideal.

Let $\mathbf{q} = (q_{ij}) \in M_N(\Bbbk)$ be a skew-symmetric matrix. The associated Poisson affine space is

$$\mathcal{A}_{\mathbf{q}} = \Bbbk[x_1, \ldots, x_N]$$

with the Poisson brackets

$$\{x_i, x_j\} = q_{ij} x_i x_j, \qquad \text{for all } i, j.$$

The corresponding Poisson torus is

$$\mathcal{T}_{\mathbf{q}} = \Bbbk[x_1^{\pm 1}, \ldots, x_N^{\pm 1}],$$

extending the Poisson bracket above.

Moment 2: statement of main results

Set-up

Henceforth,

$$R = \Bbbk[x_1]_P[x_2;\sigma_2,\delta_2]_P \cdots [x_N;\sigma_N,\delta_N]_P$$

is a PNA of rank n. Recall that R is Q-graded, where $Q = \mathbb{Z}^n$.

- $a \in R$ homogeneous, $\mathrm{wt}(a) \in Q$ denotes its weight
- $\mathrm{Z}_P(R)$ is the Poisson center of R
- $Q^\vee = \mathrm{Hom}_{\mathrm{ab}}(Q, \mathrm{Z}_P(R)) \simeq \mathrm{Z}_P(R)^n$
- $\eta \in Q^\vee$ defines $\theta_\eta \in \mathrm{Der}_P(R)$:

$$\theta_\eta(a) = \eta(\mathrm{wt}(a))a, \qquad \text{for } a \in R \text{ homogeneous}$$

- $\theta_Q = \{\theta_\eta \mid \eta \in Q^\vee\}$

$\mathrm{Der}_P(R)$

ASSUMPTIONS:

- $\mathrm{char}\,\mathbb{k} = 0$;
- the eigenvalue matrix $\mathbf{q} = (q_{i,j})$ has entries in $\mathbb{Q}$;
- no generator x_k is Poisson central;
- a technical condition on the center.

Theorem. We have the decomposition as $\mathrm{Z}_P(R)$-modules:

$$\mathrm{Der}_P(R) = \mathrm{Ham}(A) \oplus \theta_Q.$$

So any $D \in \mathrm{Der}_P(R)$ is uniquely written as $D = \mathrm{Ham}_x + \theta_\eta$, where $x \in R$, $\eta : Q \to \mathrm{Z}(R)$ is an abelian group homomorphism and $\theta_\eta(a) = \eta(\mathrm{wt}(a))a$, for any homogeneous $a \in R$.

Corollary.
$\mathrm{PH}^1(R)$ is a free $\mathrm{Z}_P(R)$-module of rank $\mathrm{rk}(Q) = n$.

More precisely, let $\{i_1, \ldots, i_n\} = \{i \mid \delta_i = 0\}$.

We get a $\mathrm{Z}_P(R)$-basis for $\mathrm{PH}^1(R)$ given by

$$D_k = \theta_{\alpha_k^*} = x_{i_k} \frac{d}{dx_{i_k}}, \quad k = 1, \ldots, n. \quad (\alpha_k = \mathrm{wt}(x_{i_k}))$$

Theorem.

- $\mathrm{Der}_P(R) = \mathrm{Ham}(R) \oplus \bigoplus_{k=1}^{n} \mathrm{Z}_P(R) D_k$.
- $\mathrm{PH}^1(R)$ is a free $\mathrm{Z}_P(R)$-module of rank n with basis $D_1, \ldots, D_n$.

Remarks

1. $R = \Bbbk[x]$ with trivial Poisson bracket. So $x \in \mathsf{Z}_P(R)$.

$$\mathrm{Der}_P(P) = R\frac{d}{dx}$$

   is not of the predicted form.

2. $R = \Bbbk[x, y, z]$ with

$$\{x, y\} = xy, \quad \{x, z\} = xz, \quad \{y, z\} = yz.$$

   Then $\mathrm{PH}^1(R)$ is free of rank 4 over $\mathsf{Z}_P(R) = \Bbbk$.

Moment 3: quantum initial clusters

Construction of quantum initial clusters

Goodearl & Yakimov produced an initial cluster $\{y_1, \dots, y_N\} \subseteq R$ based on a coloring function $\mathsf{c} : [1, N] \twoheadrightarrow [1, n]$ and two auxiliary predecessor and successor functions $\mathsf{p}, \mathsf{s} : [1, N] \to [1, N] \cup \{\infty\}$.

We obtain embeddings

$$\mathcal{A}_{\mathbf{q}} \subseteq R \subseteq \mathcal{T}_{\mathbf{q}} \subseteq \mathrm{Fract}(R),$$

where $\mathcal{A}_{\mathbf{q}} = \Bbbk_{\mathbf{q}}[y_1, \dots, y_N]$ is a Poisson affine space and $\mathcal{T}_{\mathbf{q}} = \Bbbk_{\mathbf{q}}[y_1^{\pm 1}, \dots, y_N^{\pm 1}]$ is the associated Poisson torus.

The set $Y_\infty = \{y_i \mid s(i) = \infty\}$ has cardinality $n = \mathrm{rk}(R)$ and consists of the homogeneous Poisson-prime elements of R.

Set $\mathfrak{s}_{<\infty} = \{i \in [1, N] \mid s(i) < \infty\}$, $\mathfrak{s}_\infty = \{i \in [1, N] \mid s(i) = \infty\}$.

Unfortunately, this is all there is time for but the cluster structure turns out to be essential to showing the main results.

Muitas felicidades para o nosso amigo Manolo!