Brill-Noether Theory of stable bundles on ruled surfaces

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Joint work with Laura Costa

Classification problem $\iff$ **Moduli space** $\mathcal{M}$

$$\{\text{algebraic objects}\}/\sim \quad \iff \quad \{\text{points of } \mathcal{M}\}$$

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We want to study **isomorphism classes** of vector bundles over a smooth projective variety X.

Goal: study the geometry of a moduli space that parametrizes isomorphism classes of vector bundles.

Let X be a smooth projective variety of dimension n over K, with $\overline{K} = K$ and $\mathrm{char}(K) = 0$.

Let $E \xrightarrow{p} X$ be a vector bundle of rank r over X with fixed Chern classes

$$c_i := c_i(E) \in \mathrm{H}^{2i}(X, \mathbb{Z})$$

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- For every $x \in X$, the fiber is a r-dimensional vector space: $E_x \cong K^r$.
- Locally, the bundle is trivial: $E_{|U} \cong U \times K^r$.
- The local trivializations are glued together by transition functions

$$g_{ij} : U_i \cap U_j \longrightarrow GL_r.$$

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The set of vector bundles on X with fixed invariants is unwield. We need to impose a **stability condition** on the vector bundles.

Slope-stable bundles

Slope-stability depends on a fixed **ample divisor** H on X

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-**Slope of** E:

$$\mu_H(E) := \frac{c_1(E) \cdot H^{n-1}}{\mathrm{rank}(E)}$$

-**Slope-stability**:

Definition

E is stable with respect to H (H-**stable**) if for any vector subbundle $F \hookrightarrow E$ such that $0 < rk(F) < rk(E)$ and E/F is torsion-free, we have

$$\mu_H(F) < \mu_H(E).$$

Moduli spaces

The set of rank r H-stable vector bundles over X with fixed Chern classes c_i is parametrized by a moduli scheme $M_H := M_H(r; c_1, c_2, \ldots, c_s)$.

-For the case of surfaces we have the following result:

Theorem (Gieseker, Li; O'Grady)

If $c_2 \gg 0$, then the moduli space $M_H(r; c_1, c_2)$ is a generically smooth, irreducible, quasiprojective variety of dimension

$$2rc_2 - (r-1)c_1^2 - (r^2-1)\chi\mathcal{O}_X.$$

Brill-Noether locus

-Classical Brill-Noether Theory:

If C a smooth curve, the Brill-Noether locus is defined as

$$W_d^k(C) := \{L \in \mathrm{Pic}^d(C) \mid \quad \dim(H^0(C,L)) \geq k\}.$$

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That is, the general Brill-Noether locus is defined as

$$W_H^k(r; c_1, c_2, \ldots, c_s) := \{E \in M_H(r; c_1, c_2, \ldots, c_s) \mid \dim(\mathrm{H}^0(X, E)) \geq k\}.$$

Brill-Noether locus

-Existence of general Brill-Noether locus:

Theorem (L. Costa, R. M. Miró-Roig; B. Nugent)

Assume that, for any $E \in M_H(r; c_1, c_2, \dots, c_s)$, $\mathrm{h}^i(X, E) = 0$ for $i \geq 2$. Then, for any $k \geq 0$, there exists a ***determinantal variety*** *$W_H^k(r; c_1, c_2, \dots, c_s)$ such that*

$$\mathrm{Supp}(W_H^k(r; c_1, c_2, \dots, c_s)) = \{E \in M_H(r; c_2, \dots, c_s) | \dim(\mathrm{H}^0(X, E)) \geq k\}.$$

If X is a surface, we only need that $\mathrm{h}^2(X, E) = 0$ for any $E \in M_H(r; c_1, c_2)$.

Determinantal variety

Consider

$$\varphi : F \longrightarrow G$$

a morphism of vector bundles of ranks r and s on a variety Y.

For any $U \underset{\text{open}}{\subset} Y$, the morphism

$$F_{|U} \stackrel{A}{\to} G_{|U}$$

is given by a $s \times r$ matrix A.

The vanishing of all $(k+1)\times(k+1)$ minors of A defines the k-th determinantal variety $X_k(\varphi) \subset Y$, whose underlying set is

$$\mathrm{Supp}(X_k(\varphi)) = \{\, p \in Y \mid \mathrm{rk}(\varphi_p) \leq k \,\}.$$

Brill-Noether locus

Each non-empty irreducible component of $W_H^k := W_H^k(r; c_1, c_2, \dots, c_s)$ has dimension at least

$$\rho_H^k(r; c_1, c_2, \dots, c_s) := \dim(M_H) - k(k - \chi(r; c_1, c_2, \dots, c_s)),$$

which is called the **expected dimension**.

The different Brill-Noether loci form a filtration of M_H:

$$M_H \supseteq W_H^1 \supseteq \cdots \supseteq W_H^k \supseteq W_H^{k+1} \supseteq \cdots$$

Brill-Noether locus

In the context of the Brill-Noether Theory, one can immediately raise the following natural questions:

Questions

1) Is $W_H^k(r; c_1, c_2, \dots, c_s)$ empty or non-empty?
2) Does $\rho_H^k(r; c_1, c_2) < 0$ imply that $W_H^k(r; c_1, c_2, \dots, c_s) = \emptyset$?
3) Is $W_H^k(r; c_1, c_2, \dots, c_s)$ smooth?
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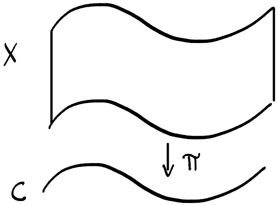
Ruled surfaces

A ruled surface is a smooth projective surface X, together with a surjective map $\pi : X \to C$ to a nonsingular curve C of genus $g \geq 0$.



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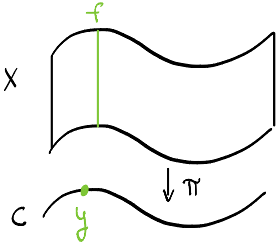

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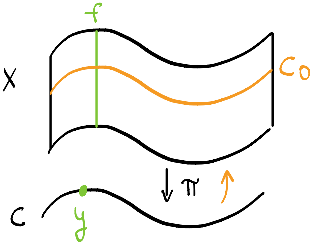

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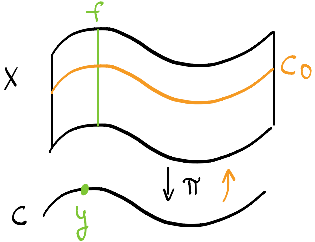

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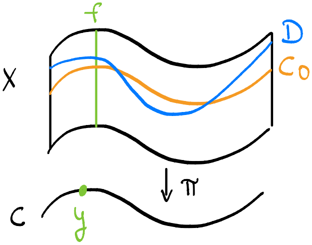
$X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank 2 normalized bundle on C. If $\mathfrak{e}$ is the divisor on C corresponding to $\wedge^2\mathcal{E}$, $e := -\deg(\mathfrak{e})$ is an invariant of X.

From now on, we will assume that $e \geq 0$.

Ruled surfaces



$$\mathrm{Pic}(X) \cong \mathbb{Z}C_0 \oplus \pi^* \mathrm{Pic}(C)$$

$$\mathrm{Num}(X) \cong \mathbb{Z} \oplus \mathbb{Z},$$

generated by C_0 and f such that $C_0 \cdot f = 1$, $C_0^2 = -e$ and $f^2 = 0$.

The class of a divisor D can be written as $D \equiv aS + bf$, where $b = \deg(\mathfrak{b})$ for some $\mathfrak{b} \in \mathrm{Pic}(C)$.

-Ample divisors for $e \geq 0$:

$H \equiv \alpha C_0 + \beta f$ is ample if and only if $\alpha > 0$ and $\beta > \alpha e$.

(1) Is $W_H^k(2; c_1, c_2)$ empty or non-empty?

Non-emptiness of BN locus

We study $W_H^k(2; c_1, c_2)$ for $c_1 \in \{0, C_0, C_0 + f, f\}$ and $c_2 \gg 0$.

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-Existence of the Brill-Noether locus:

We check that, for any $E \in M_H(2; c_1, c_2)$, $\mathrm{h}^2(X, E) = 0$.

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-Existence of the Brill-Noether locus:

We check that, for any $E \in M_H(2; c_1, c_2)$, $\mathrm{h}^2(X, E) = 0$.

Case $c_1 = 0$:

We prove that $W_H^k(2; 0, c_2) = \emptyset$ for any ample divisor H.

Non-emptiness of BN locus

Case $c_1 \in \{C_0, C_0 + f\}$**:**

Consider the family of ample divisors H on X such that $H \equiv \alpha C_0 + \beta f$ with

$$\alpha > 0 \quad \text{and} \quad \max\{\alpha(8 + e - m + 2g), \alpha(6g - 4 + e - m)\} < \beta,$$

where $m \in \{0, 1\}$.

Theorem (L. Costa, I. Macías Tarrío)

The Brill-Noether locus $W_H^k(2; c_1, c_2) \neq \emptyset$ for any k in the range

$$1 \le k < \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)].$$

Idea of the proof

We consider the family $\mathcal{G}$ of rank two vector bundles E on X given by a non-trivial extension

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E \to I_Z(C_0 + (\mathfrak{m} - \mathfrak{b})f) \to 0,$$

where $\mathfrak{b} \in \mathrm{Pic}(C)$ of degree

$$b = \deg(\mathfrak{b}) = k - 1 + g,$$

where

$$\max\{1, g\} \leq k < \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)]$$

and Z is a 0-dimensional generic subscheme of length $|Z| = c_2 - b$.

We have proved that, for $c_1 \in \{C_0, C_0 + f\}$, $W_H^k(2; c_1, c_2) \neq \emptyset$ for

$$1 \leq k < \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)].$$

Hence, we get:

$$M_H \supseteq W_H^1 \supseteq W_H^2 \supseteq \cdots \supseteq W_H^k \neq \emptyset$$

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Is the bound of k sharp?

$$M_H \supseteq W_H^1 \supseteq \cdots \supseteq W_H^k \supseteq W_H^{k+1} \supseteq \cdots$$

Emptiness of BN

The bound is sharp!

Theorem (L. Costa, I. Macías Tarrío)

$W_H^k(2; c_1, c_2) = \emptyset$ *for all* $k \geq \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)]$.

$$M_H \supseteq W_H^1 \supseteq \quad W_H^2 \supseteq \quad \cdots \supseteq \quad \begin{array}{c} W_H^k \\ \nparallel \\ \emptyset \end{array} \quad \supseteq W_H^{k+1} = \emptyset$$

Hence,

Corollary (L. Costa, I. Macías Tarrío)

$W_H^k(2; c_1, c_2) \neq \emptyset \Leftrightarrow 1 \leq k < \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)]$.

Another non-emptiness result

Case $c_1 = f$:

Let $H \equiv C_0 + \beta f$ be an ample divisor on X.

Theorem (L. Costa, I. Macías Tarrío)

$W_H^k(2; f, c_2) \neq \emptyset$ if and only if $k = 1, 2$.

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