Rota-Baxter type Operators on Trusses and Derived Structures

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Outline

Based on the following joint paper and the references therein:

T. Chtioui, M. Elhamdadi, S. Mabrouk and A. Makhlouf, *Rota-Baxter type operators on trusses and derived structures*, arXiv 2025.

Basics about trusses:

T. Brzeziński, *Trusses: between braces and rings*, Trans. Amer. Math. Soc. 372, no. 6 (2019)
T. Brzeziński, *Trusses: Paragons, ideals and modules*, J. Pure Appl. Algebra **224** (2020).

Heaps, trusses, groups, rings ...

Heaps were already considered first in:

• H. Prüfer, Theorie der Abelschen Gruppen. I. Grundeigenschaften, Math. Z. 20 (1924), 165–187.

• R. Baer, Zur Einfhrung des Scharbegriffs, J. Reine Angew. Math. 160 (1929), 199–207.

Heaps, trusses, groups, rings ...

Definition (Abelian Heap)

An abelian heap is a set H equipped with a ternary operation $[\cdot,\cdot,\cdot]$ satisfying, for all $a, b, c, x, y \in H$, the following identities:

$$[[a,b,c],x,y] = [a,b,[c,x,y]], \tag{1}$$

$$[a,a,b] = [b,a,a] = b, \tag{2}$$

$$[a,b,c] = [c,b,a]. \tag{3}$$

Eq.(1) represents the *associativity law* for heaps,
Eq.(2) are known as the *Mal'cev identities*,
Eq.(3) expresses the *commutativity law* for heaps.

Heaps vs Groups

- For any $e \in H$, the set H equipped with the binary operation

$$+_e = [-, e, -]$$

forms an abelian group, known as the **retract** of H.
The chosen element e serves as the identity element of this group, and the inverse $-_e a$ of an element $a \in H$ is given by:

$$-_e a = [e, a, e].$$

We denote this unique (up to isomorphism) group by $\mathcal{G}(H; e)$.
- Conversely, for any abelian group G, the operation

$$[a, b, c] = a - b + c$$

defines a heap structure on G, which we denote by $\mathcal{H}(G)$.

A *homomorphism of heaps* is a map that preserves the heap operation.

In particular, any group homomorphism is also a heap homomorphism for the corresponding heap.

Thus, the assignment $\mathcal{H} : G \to \mathcal{H}(G)$ defines a functor from the category of abelian groups to the category of abelian heaps.

Examples

1. Any singleton set, say $\{*\}$, with: $[*,*,*]=*$. This heap serves as the terminal object in the category of heaps, meaning there is a unique morphism from any heap to $\{*\}$.
2. The set $H=2\mathbb{Z}+1$ of all odd integers, with the ternary operation:

$$[2n+1, 2m+1, 2p+1] = 2(n-m+p)+1,$$

Furthermore, if we fix the element $1 \in H$, the set H can be retracted to a group structure with :

$$(2m+1)\cdot(2n+1) = 2(m+n)+1.$$

It is isomorphic to the additive group of integers $\mathbb{Z}$ via $2m+1 \mapsto m$.

Truss

Definition

A Truss is an abelian heap $(T, [\cdot,\cdot,\cdot])$ *equipped with a binary operation, denoted by* $\cdot$ *or just by juxtaposition and called multiplication, such that* $(T, \cdot)$ *is a semigroup and for all* $a, b, c, d \in T$*, we have*

$$a[b,c,d] = [ab, ac, ad] \quad \textit{and} \quad [a,b,c]d = [ad, bd, cd]. \tag{4}$$

A *morphism of trusses* is a map that is a homomorphism of both heaps and semigroups.
A subtruss S of a truss $(T, [\cdot,\cdot,\cdot], \cdot)$ is subheap such that $S \cdot S \subset S$.

Examples (Brzeziński 2020) (1)

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss and let X be a set. The set T^X of all functions $X \to T$ is a truss with the pointwise defined operations, i.e. for all $f, g, h \in T^X$,

$$[f, g, h] : X \to T, \qquad x \mapsto [f(x), g(x), h(x)], \tag{5}$$

$$fg : X \to T, \qquad x \mapsto f(x)g(x). \tag{6}$$

Examples (2)

Consider the Klein four group (additive) $V_4 = \{0, a, b, a+b\}$.

Then V_4 becomes a truss when equipped with the heap operation $[\cdot,\cdot,\cdot]_+$ (induced by the group addition) and the multiplication defined by the table

$\cdot$	0	a	b	$a+b$
0	0	0	b	a
a	a	a	$a+b$	0
b	b	b	0	$a+b$
$a+b$	$a+b$	$a+b$	$a+b$	$a\,b$

Examples (3)

Let R be a ring and $e \in M_n(R)$ be an $n \times n$ idempotent matrix (that is, $e^2 = e$). Consider the module R^n, which forms an abelian group under addition. This additive structure induces the natural heap operation defined by

$$[x, y, z] = x - y + z, \quad \text{for all } x, y, z \in R^n.$$

Now, define a binary multiplication on R^n by

$$(r_1, \ldots, r_n) \cdot (s_1, \ldots, s_n) = (r_1, \ldots, r_n) + (s_1, \ldots, s_n)e.$$

Then $(R^n, [\cdot, \cdot, \cdot], \cdot)$ is a truss.

Non-commutative truss structures on $\mathbb{Z}$ (Brzeziński 2020)

The set of integers $\mathbb{Z}$ with the usual addition can be viewed as a heap with induced operation,

$$[l, m, n]_+ = l - m + n. \tag{7}$$

There are two non-commutative truss structures on $(\mathbb{Z}, [\cdot, \cdot, \cdot]_+)$ with products defined for all $m, n \in \mathbb{Z}$ by

$$m \cdot n = m \quad \text{or} \quad m \cdot n = n. \tag{8}$$

Commutative truss structures on $\mathbb{Z}$ (Brzeziński 2020)

Up to isomorphism, the following commutative products equip $(\mathbb{Z}, [\cdot,\cdot,\cdot]_+)$ with different truss structures, for all $m, n \in \mathbb{Z}$:

1. For all $a \in \mathbb{N}$,

$$m \cdot n = amn, \quad a \neq 1, \tag{9}$$

$$m \cdot n = amn + m + n. \tag{10}$$

2. For all $c \in \mathbb{N} \setminus \{0\}$,

$$m \cdot n = c, \tag{11}$$

$$m \cdot n = m + n + c. \tag{12}$$

3. For all $a \in \mathbb{N} \setminus \{0, 1, 2\}$, $b \in \{2, 3, \ldots, a-1\}$, and $c \in \mathbb{N} \setminus \{0\}$ such that $ac = b(b-1)$,

$$m \cdot n = amn + b(m+n) + c. \tag{13}$$

Classification of trusses on $\mathbb{Z}_2$

Any truss product on $\mathbb{Z}_2$, with the standard abelian heap ternary bracket $[\cdot,\cdot,\cdot]_+$, is isomorphic to a truss corresponding to one of the binary operations given by the following Cayley tables providing the multiplication of the ith row elements by the jth column elements.

$$(1) := \begin{array}{|c|c|c|}\hline \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array},\quad (2) := \begin{array}{|c|c|c|}\hline \cdot & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 0 \\ \hline \end{array},\quad (3) := \begin{array}{|c|c|c|}\hline \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array},\quad (4) := \begin{array}{|c|c|c|}\hline \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline \end{array},\quad (5) := \begin{array}{|c|c|c|}\hline \cdot & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 1 & 0 & 1 \\ \hline \end{array}.$$

Notice that the structures given by the tables:

$$(6) := \begin{array}{|c|c|c|}\hline \cdot & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array},\quad (7) := \begin{array}{|c|c|c|}\hline \cdot & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array},\quad (8) := \begin{array}{|c|c|c|}\hline \cdot & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

are isomorphic to (2), (3) and (1), respectively.

Absorber element of a truss

Definition

A left (resp. right) absorber element of a truss T is an element $\mathbf{0} \in T$ such that, for all $t \in T$, $t \cdot \mathbf{0} = \mathbf{0}$ (resp. $\mathbf{0} \cdot t = \mathbf{0}$). We say that $\mathbf{0}$ is an absorber element if it is a left and right absorber element.

Rings vs Trusses

By a *ring* we mean an associative ring that is not necessarily unital.

Given a ring $(R, +, \cdot)$, we can associate a truss $\mathcal{T}(R)$ where the heap structure is given by $\mathcal{H}(R, +)$, while the multiplication remains unchanged.

The assignment $\mathcal{T} : R \to \mathcal{T}(R)$, acting as the identity on morphisms, defines a functor from the category of rings to the category of trusses.

Conversely, if a truss T possesses an absorber element $\mathbf{0} \in T$. Then the structure $\mathcal{R}(T;\mathbf{0}) := (\mathcal{G}(T;\mathbf{0}), \cdot)$ forms a ring. Here, $\mathcal{G}(T;\mathbf{0})$ denotes the group-like structure derived from the truss T with respect to the absorber $\mathbf{0}$, and the multiplication operation is preserved.
Moreover, this construction satisfies the identity:

$$T = \mathcal{T}(\mathcal{R}(T;\mathbf{0})).$$

This identity shows that the original truss T can be recovered by applying the functor $\mathcal{T}$ to the ring $\mathcal{R}(T;\mathbf{0})$, establishing a natural correspondence between trusses with absorbers and rings.

Lemma

In an abelian heap H we have, for all $x_i, y_i, z_i \in H$,

$$[[x_1, x_2, x_3], [y_1, y_2, y_3], [z_1, z_2, z_3]] = [[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]]. \tag{14}$$

Proposition

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss. Then the set $T \times T$ has a truss structure given by

$$\{(a, x), (b, y), (c, y)\} = ([a, b, c], [x, y, z]), \tag{15}$$

$$(a, x) \odot (b, y) = (a \cdot b, [a \cdot y, \mathbf{0}, x \cdot b]), \tag{16}$$

for any $a, b, c, x, y, z \in T$. This truss is denoted by $T \ltimes T$.

More generally, let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss and e be an element in the center (i.e. $e \cdot x = x \cdot e$ for all x).
Define a truss product by

$$(a, x) \boxdot_e (b, y) = (a \cdot b, [a \cdot y, e \cdot x \cdot y, x \cdot b]). \tag{17}$$

Proposition

Under the above notations, $(T \times T, [\cdot,\cdot,\cdot], \boxdot_e)$ is a truss with the heap structure given in (17), which is denoted by $T \bowtie_e T$.

Rota-Baxter operators on trusses

We assume that all trusses are equipped with absorber elements, denoted by $\mathbf{0}$.

Definition

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss. A map $R : T \to T$ is called a Rota-Baxter operator of weight $\mathbf{0}$, where $\mathbf{0}$ is an absorber element of T, if it is a heap morphism such that $R(\mathbf{0}) = \mathbf{0}$ and satisfying

$$R(x) \cdot R(y) = R[R(x) \cdot y, \mathbf{0}, x \cdot R(y)], \quad \forall x, y \in T. \tag{18}$$

Example

Consider the abelian heap $(\mathbb{Z}, [\cdot,\cdot,\cdot]_+)$ and let $a \in \mathbb{Z}$. The binary operation defined by,

$$m \cdot_a n = a, \quad \forall m, n \in \mathbb{Z}$$

makes $(\mathbb{Z}, [\cdot,\cdot,\cdot]_+)$ into a truss, in which a is an absorber element. Define the map $R : \mathbb{Z} \to \mathbb{Z}$ by

$$R(m) = [m, a, m]_+, \quad \forall m \in \mathbb{Z}.$$

Then R is a Rota-Baxter operator on $(\mathbb{Z}, [\cdot,\cdot,\cdot]_+, \cdot_a)$.

Derivations and Rota-Baxter operators

Brzeziński (2020) introduced a concept of a derivation for trusses. We introduce a new definition of a truss derivation that fits with Rota-Baxter structures.

Definition

A derivation on a truss $(T, [\cdot,\cdot,\cdot], \cdot)$ is a map $D : T \to T$ which is both a heap's morphism and satisfying

$$D(\mathbf{0}) = \mathbf{0}, \tag{19}$$

$$D(x \cdot y) = [D(x) \cdot y, \mathbf{0}, x \cdot D(y)], \quad \forall x, y \in T. \tag{20}$$

Proposition

If a derivation D of a truss is bijective, then D^{-1} is a Rota-Baxter operator and vice-versa.

Proposition

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss, the map $R : T \to T$ is a Rota-Baxter operator of weight $\mathbf{0}$ if and only if its graph

$$\mathrm{Gr}(R) := \{(R(x), x),\ x \in T\}$$

is a sub-truss of $T \ltimes T$.

Theorem

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss and $R : T \to T$ be a Rota-Baxter operator of weight $\mathbf{0}$. Define the product

$$x \diamond y = [R(x) \cdot y, \mathbf{0}, x \cdot R(y)]. \tag{21}$$

Then $(T, [\cdot,\cdot,\cdot], \diamond)$ is also a truss with the same absorber element.

Dendriform trusses: a definition

A dendriform truss is a tuple $(T, [\cdot,\cdot,\cdot], \prec, \succ, \mathbf{0})$, where $(T, [\cdot,\cdot,\cdot])$ is an abelian heap, $\mathbf{0} \in T$ is a distinguished element and $\prec, \succ : T \otimes T \to T$ are two binary products satisfying:

$$x \succ \mathbf{0} = \mathbf{0} \succ x = \mathbf{0} \prec x = x \prec \mathbf{0} = \mathbf{0}, \tag{22}$$

$$[x, y, z] \succ t = [x \succ t, y \succ t, z \succ t], \ x \prec [y, z, t] = [x \prec y, x \prec z, x \prec t], \tag{23}$$

$$[x \succ y, \mathbf{0}, x \prec y] \succ z = x \succ (y \succ z), \tag{24}$$

$$(x \succ y) \prec z = x \succ (y \prec z), \tag{25}$$

$$(x \prec y) \prec z = x \prec [y \succ z, \mathbf{0}, y \prec z], \tag{26}$$

for all $x, y, z \in T$. Similarly to trusses, the element $\mathbf{0}$ is called the absorber element of the dendriform truss T.

Theorem

Let $(T, [\cdot,\cdot,\cdot], \prec, \succ, \mathbf{0})$ *be a dendriform truss.*
Then $(T, [\cdot,\cdot,\cdot], \diamond)$ *is a truss, where*

$$x \diamond y = [x \succ y, \mathbf{0}, x \prec y]. \tag{27}$$

The tuple $(T, [\cdot,\cdot,\cdot], \diamond)$ *is called the subadjacent truss of* $(T, [\cdot,\cdot,\cdot], \prec, \succ, \mathbf{0})$*.*

Conversely, if we have a truss T, we can get a dendriform truss on the same heap structure of T by means of a Rota-Baxter operator.

Theorem

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss and $R : T \to T$ be a Rota-Baxter operator of weight $\mathbf{0}$. Then the binary products

$$x \succ y = R(x) \cdot y, \quad x \prec y = x \cdot R(y) \tag{28}$$

provide a dendriform truss structure on T with absorber element $\mathbf{0}$. In addition, R is a truss morphism.

Dendriform Rings

Definition

Let $(A, +, 0)$ be an abelian group and $\succ, \prec : A \otimes A \to A$ be two binary products. The triple $(A, +, \succ, \prec)$ is called a dendriform ring if:

$$
\begin{aligned}
& x \succ 0 = 0 \prec x = 0, \\
& (x + y) \succ z = x \succ z + y \succ z, \quad x \prec (y + z) = x \prec y + x \prec z, \\
& (x \succ y + x \prec y) \succ z = x \succ (y \succ z), \\
& (x \succ y) \prec z = x \succ (y \prec z), \\
& (x \prec y) \prec z = x \prec (y \succ z + y \prec z),
\end{aligned}
$$

for all $x, y, z \in A$.

Remark

Note that if $(A, +, \succ, \prec)$ is a dendriform ring, then $(A, +, \times)$ is a ring called the subadjacent ring, where

$$x \times y = x \succ y + x \prec y, \quad \forall x, y \in A.$$

Theorem

There is a one-to-one correspondence between dendriform rings and dendriform trusses.

Rota-Baxter operators on rings

Definition

Let $(A, +, \times)$ be a ring and $R : A \to A$ be a group homomorphism. We say that R is a Rota-Baxter operator on A if it satisfies the following identity, for all $x, y \in A$,

$$R(x) \times R(y) = R(R(x) \times y + x \times R(y)), \tag{29}$$

Proposition

Let $(A, +, \times)$ be a ring and $R : A \to A$ be a Rota-Baxter operator. Then the following products

$$x \succ y = R(x) \times y \quad \text{and} \quad x \prec y = x \times R(y), \quad \forall x, y \in A, \tag{30}$$

provide a dendriform ring structure on A.

Proposition

Let $(A, +, \times)$ be a ring and $R : A \to A$ be a Rota-Baxter operator on A. Then R is a Rota-Baxter operator on the associated truss $T(A)$.
Conversely, If R is a Rota-Baxter operator on a truss T, then it is still a Rota-Baxter operator on its associated ring.

Corollary

Let $(A, +, \succ, \prec)$ *be a dendriform ring and define*

$$[x, y, z] = x - y + z,\ \forall x, y, z \in A.$$

Then $(T, [\cdot, \cdot, \cdot], \succ, \prec)$ *is a dendriform truss.*

Corollary

Let $(T, [\cdot, \cdot, \cdot], \succ, \prec)$ *be a dendriform truss. Suppose that there is an absorber element* $a \in T$ *and define the operation*

$$x + y = [x, a, y],\ \forall x, y, \in A.$$

Then $(A, +, \succ, \prec)$ *is a dendriform ring.*

Weighted Rota-Baxter operators on trusses

Definition

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss and $a \in T$ be an element in the center with respect to the binary operation. A heap homomorphism $R : T \to T$ is called a weighted Rota-Baxter of weight a on T if it satisfies

$$R(x) \cdot R(y) = R([R(x) \cdot y, a \cdot x \cdot y, x \cdot R(y)] \tag{31}$$

for all $x, y \in T$.

This definition is in the same spirit as the original definition of Rota-Baxter operators given by F. V. Atkinson, see also more recent observations and developments by Goncharov.

• If the truss admits an absorber element $\mathbf{0}$, we recover for $a = \mathbf{0}$ the Rota-Baxter operators of weight $\mathbf{0}$.

• If the truss is unital with 1 as a unit element, setting $a = 1$, one gets Rota-Baxter operator of weight 1.

Definition

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss (not necessarily unital). We say that a heap homomorphism $R : T \to T$ is a Rota-Baxter operator of weight 1 on T if it satisfies

$$R(x) \cdot R(y) = R([R(x) \cdot y, x \cdot y, x \cdot R(y)]. \qquad (32)$$

Example

Let $\mathbb{Z}_2$ be the truss with the heap operation $[\cdot,\cdot,\cdot]_+$ and multiplication given by (3) $:=$

$\cdot$	0	1
0	0	0
1	0	1

.

Define the map $R : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $R(1) = 0$ and $R(0) = 1$. Then R is a Rota-Baxter operator of weight 1 on $\mathbb{Z}_2$.

Proposition

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss, the map $R : T \to T$ is a weighted Rota-Baxter operator of weight a if and only if its graph

$$\mathrm{Gr}(R) := \{(R(x), x),\ x \in T\}$$

is a sub-truss of $T \bowtie_a T$.

Tridendriform trusses

Definition

Let $(T, [\cdot,\cdot,\cdot])$ *be an abelian heap and* $\succ, \vee, \prec : T \otimes T \to T$ *be three binary operations which are distributive with respect to the heap structure. We say that* $(T, [\cdot,\cdot,\cdot], \succ, \vee, \prec)$ *is a* **tridendriform truss** *if it satisfies the following set of identities:*

$$(x \prec y) \prec z = x \prec [y \succ z, y \vee z, y \prec z], \tag{33}$$

$$(x \succ y) \prec z = x \succ (y \prec z), \tag{34}$$

$$[x \succ y, x \vee y, x \prec y] \succ z = x \succ (y \succ z), \tag{35}$$

$$(x \succ y) \vee z = x \succ (y \vee z), \tag{36}$$

$$(x \prec y) \vee z = x \vee (y \succ z), \tag{37}$$

$$(x \vee y) \prec z = x \vee (y \prec z), \tag{38}$$

$$(x \vee y) \vee z = x \vee (y \vee z), \tag{39}$$

for all $x, y, z \in T$.

Proposition

Let $(T, [\cdot,\cdot,\cdot], \succ, \vee, \prec)$ be a tridendriform truss. Then $(T, [\cdot,\cdot,\cdot], \diamond)$ is a truss, where

$$x \diamond y = [x \succ y, x \vee y, x \prec y], \quad \forall x, y \in T. \tag{40}$$

Theorem

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss and $R : T \to T$ be a weighted Rota-Baxter operator of weight a. Define the products

$$x \succ y = R(x) \cdot y, \quad x \prec y = x \cdot R(y), \quad x \vee y = a \cdot x \cdot y, \ \forall x, y \in T. \tag{41}$$

Then $(T, [\cdot,\cdot,\cdot], \succ, \vee, \prec)$ is a tridendriform truss.
In addition, if $(T, [\cdot,\cdot,\cdot], \diamond)$ is the subadjacent truss, R is a truss homomorphism from $(T, [\cdot,\cdot,\cdot], \diamond)$ to $(T, [\cdot,\cdot,\cdot], \cdot)$.

Example

Consider the truss structure on $\mathbb{Z}_2$ given by table (3) and the Rota-Baxter operator given in the previous Example. Then, according to previous Theorem, the binary operations given by

$\vee$	0	1
0	0	0
1	0	1

,

$\succ$	0	1
0	0	1
1	0	0

and

$\prec$	0	1
0	0	0
1	1	0

endow the abelian heap $\mathbb{Z}_2$ with a tridendriform truss structure.

Tridendriform trusses and tridendriform rings

Definition

A tridendriform ring is a quardruple $(A, \prec, \succ, \vee)$ consisting of three binary operations $\succ, \prec, \vee : A \times A \to A$, which satisfy the following axioms:

$$\begin{aligned}
(a \prec b) \prec c &= a \prec (b * c),\\
(a \succ b) \prec c &= a \succ (b \prec c),\\
(a * b) \succ c &= a \succ (b \succ c),\\
(a \succ b) \vee c &= a \succ (b \vee c),\\
(a \prec b) \vee c &= a \vee (b \succ c),\\
(a \vee b) \prec c &= a \vee (b \prec c),\\
(a \vee b) \vee c &= a \vee (b \vee c),
\end{aligned} \tag{42}$$

where $* = \succ - \vee + \prec$, for $a, b, c \in A$.

Corollary

Let $(A, +, \succ, \vee, \prec)$ *be a tridendriform ring and define*

$$[x, y, z] = x - y + z, \ \forall x, y, z \in A.$$

Then $(T, [\cdot, \cdot, \cdot], \succ, \vee, \prec)$ *is a tridendriform truss.*

Corollary

Let $(T, [\cdot, \cdot, \cdot], \succ, \vee, \prec)$ *be a tridendriform truss. Suppose that there is an absorber element* $a \in T$ *and define the operation*

$$x + y = [x, a, y], \ \forall x, y, \in A.$$

Then $(A, +, \succ, \vee, \prec)$ *is a tridendriform ring.*

Reynolds and Nijenhuis operators, NS-rings and NS-trusses.

Definition

An NS-ring is an abelian group $(A,+)$ together with three binary operations $\succ, \prec, \curlyvee : A \otimes A \to A$ that are distributive with respect to addition and satisfying the following identities:

$$(x \prec y) \prec z = x \prec (y \bullet z), \tag{43}$$

$$(x \succ y) \prec z = x \succ (y \prec z), \tag{44}$$

$$(x \bullet y) \succ z = x \succ (y \succ z), \tag{45}$$

$$(x \curlyvee y) \prec z + (x \bullet y) \curlyvee z = x \succ (y \curlyvee z) + x \curlyvee (y \bullet z), \tag{46}$$

for all $x, y, z \in A$ and where $x \bullet y = x \succ y + x \prec z + x \curlyvee y$.
We will denote such a dendriform ring by $(A, \prec, \curlyvee, \prec)$.

Remark

Exactly as for associative algebras, we can check that $(A, +, \bullet)$ *is also a ring. Moreover, if the binary operation* $\curlyvee$ *is trivial, then the tuple* $(A, +, \succ, \prec)$ *is a dendriform ring provided that the absorption axiom* (22) *holds.*

Now, we generalize the notion of NS-rings to the truss-setting.

Definition

An **NS-truss** *is an abelian heap* $(T, [\cdot,\cdot,\cdot])$ *together with three binary operations* $\succ, \prec$ *and* $\curlyvee : T \otimes T \to T$ *satisfying the following axioms:*

$$(x \prec y) \prec z = x \prec [y \succ z, y \curlyvee z, y \prec z], \tag{47}$$

$$(x \succ y) \prec z = x \succ (y \prec z), \tag{48}$$

$$[x \succ y, x \curlyvee y, x \prec y] \succ z = x \succ (y \succ z), \tag{49}$$

$$(x \curlyvee y) \prec z = x \succ (y \curlyvee z), \tag{50}$$

$$[x \succ y, x \curlyvee y, x \prec y] \curlyvee z = x \curlyvee [y \succ z, y \curlyvee z, y \prec z], \tag{51}$$

for all $x, y, z \in T$.
This dendriform truss will be denoted by $(T, [\cdot,\cdot,\cdot], \succ, \curlyvee, \prec)$.

Theorem

Let $(T, [\cdot,\cdot,\cdot], \succ, \curlyvee, \prec)$ *be an NS-truss. Then the binary product*

$$x \diamond y = [x \succ y, x \curlyvee y, x \prec y], \quad x, y \in T, \tag{52}$$

provides a truss structure on the abelian heap $(T, [\cdot,\cdot,\cdot])$.

Reynolds operators on trusses

Such operators allow us to split a truss multiplication into three compatible parts which give an NS-truss structure.

Definition

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss. A map $P : T \to T$ is called a Reynolds operator if it is a heap morphism of $(T, [\cdot,\cdot,\cdot])$ and satisfying

$$P(x)\cdot P(y) = P[P(x)\cdot y, P(x)\cdot P(y), x\cdot P(y)], \quad \forall x, y \in T. \quad (53)$$

Modified derivation and Reynolds operator

The definition of a derivation on a truss given in Brzezinski (2022) is as follows. We call it a modified derivation.

A **modified derivation** on a truss $(T, [\cdot,\cdot,\cdot], \cdot)$ is a heap morphism $D : T \to T$ which satisfies the following Leibniz-like rule:

$$D(x \cdot y) = [x \cdot D(y), x \cdot y, D(x) \cdot y], \forall x, y \in T. \tag{54}$$

Then we have the following equivalence:

D is a modified derivation if and only if D^{-1} is a Reynolds operator.

Theorem

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss and $P : T \to T$ be a Reynolds operator. Define a product

$$x \circ y = [P(x) \cdot y, P(x) \cdot P(y), x \cdot P(y)]. \tag{55}$$

Then $(T, [\cdot,\cdot,\cdot], \circ)$ is also a truss.

Theorem

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss and $P : T \to T$ be a Reynolds operator. Define the products

$$x \succ y = P(x) \cdot y, \quad x \curlyvee y = P(x) \cdot P(y), \quad x \prec y = x \cdot P(y), \ \forall x, y \in T \tag{56}$$

Then $(T, [\cdot,\cdot,\cdot], \succ, \curlyvee, \prec)$ is an NS-truss.
In addition if $(T, [\cdot,\cdot,\cdot], \diamond)$ is the subadjacent truss, then P is a truss homomorphism from $(T, [\cdot,\cdot,\cdot], \diamond)$ to $(T, [\cdot,\cdot,\cdot], \cdot)$.

Nijenhuis operators on trusses

The notion of Nijenhuis operator on a truss was introduced in Brzezinski (2022). We provide a connection with NS-trusses.

Definition

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss. A heap homomorphism $N : T \to T$ is called a Nijenhuis operator if it satisfies the following integrability condition:

$$N(x) \cdot N(y) = N([N(x) \cdot y, N(x \cdot y), x \cdot N(y)]), \tag{57}$$

for all $x, y \in T$.

Lemma

Let $(T, [\cdot, \cdot, \cdot], \cdot)$ be a truss and $N : T \to T$ be a Nijenhuis operator. Then the product

$$x \circ y = [N(x) \cdot y, N(x \cdot y), x \cdot N(y)] \tag{58}$$

provides a truss structure on T.
Moreover, N is a truss morphism from $(T, [\cdot, \cdot, \cdot], \circ)$ to $(T, [\cdot, \cdot, \cdot], \cdot)$.

Proposition

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ *be a truss and* $R : T \to T$ *be a Rota-Baxter operator of weight* $\mathbf{0}$*.*
Define an operator on $T \times T$ *by*

$$N(x,y) = (R(y), \mathbf{0}), \quad \forall x, y \in T.$$

Then N *is Nijenhuis operator on the truss* $T \ltimes T$*.*

Proposition

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss and $N : T \to T$ be a Nijenhuis operator. Then the binary operations

$$x \succ y = N(x) \cdot y, \quad x \curlyvee y = N(x \cdot y), \quad x \prec y = x \cdot N(y) \quad (59)$$

provide an NS-truss structure on T.

We have shown that ring, trusses, dendriform rings, dendriform trusses, NS-rings and NS-trusses are closely related in the sense of commutative diagram of categories as follows:

$$\begin{tikzcd}
\text{NS-Rings} \arrow[r, "\text{Theorem 15}"] \arrow[dd, bend right, "\text{Remark 2}"'] & \text{NS-Trusses.} \\
\text{Dend. Rings} \arrow[u] \arrow[r] & \text{Dend. Trusses} \arrow[u] \arrow[d, <->, "\text{Theo.6}"', "\text{Theo. 5}"] \\
\text{Rings.} \arrow[u] \arrow[r, <->, "\text{Remark .}"] & \text{Trusses} \arrow[uu, bend right]
\end{tikzcd}$$

Averaging operators on trusses, di-trusses and tri-trusses

Definition

A **di-truss** $(T, [\cdot,\cdot,\cdot], \vdash, \dashv)$ *consists of an abelian heap* $(A, [\cdot,\cdot,\cdot])$ *equipped with two associative products* $\vdash, \dashv$: $A \otimes A \to A$ *which are distributive with respect to bracket and satisfy*

$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{60}$$

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{61}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{62}$$

for all $x, y, z \in A$.

Definition

A di-truss $(A, [\cdot,\cdot,\cdot], \vdash, \dashv)$ is called **tri-truss** *if it is equipped with an associative product $\perp : A \otimes A \to A$ which is distributive with respect to the bracket and satisfy*

$$(x \dashv y) \dashv z = x \dashv (y \perp z), \tag{63}$$
$$(x \perp y) \dashv z = x \perp (y \dashv z), \tag{64}$$
$$(x \dashv y) \perp z = x \perp (y \vdash z), \tag{65}$$
$$(x \vdash y) \perp z = x \vdash (y \perp z), \tag{66}$$
$$(x \perp y) \vdash z = x \vdash (y \vdash z), \tag{67}$$

for all $x, y, z \in A$.

Definition

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss. A map $\mathcal{K} : T \to T$ is called

1. **left averaging operator** *if it is a heap morphism of $(T, [\cdot,\cdot,\cdot])$ and satisfying*
$$\mathcal{K}(x) \cdot \mathcal{K}(y) = \mathcal{K}(\mathcal{K}(x) \cdot y), \quad \forall x, y \in T. \tag{68}$$
2. **right averaging operator** *if it is a heap morphism of $(T, [\cdot,\cdot,\cdot])$ and satisfying*
$$\mathcal{K}(x) \cdot \mathcal{K}(y) = \mathcal{K}(x \cdot \mathcal{K}(y)), \quad \forall x, y \in T. \tag{69}$$
3. **averaging operator** *if it is both left and right averaging.*
4. **homomorphic averaging operator** *if it is both a truss homomorphism and averaging operator.*

Let $(T, [\cdot,\cdot,\cdot], \cdot)$ be a truss. Define on $T \times T$ the operations

$$[[(a,x),(b,y),(c,y)]] = ([a,b,c],[x,y,z]), \tag{70}$$
$$(a,x) \triangleright (b,y) = (a \cdot b, a \cdot y), \tag{71}$$
$$(a,x) \triangleleft (b,y) = (a \cdot b, x \cdot b), \tag{72}$$
$$(a,x) \triangle (b,y) = (a \cdot b, x \cdot y), \tag{73}$$

for all $a,b,c,x,y,z \in T$. Then $(T \times T, [[\cdot,\cdot,\cdot]], \triangleright)$ (resp. $(T \times T, [[\cdot,\cdot,\cdot]], \triangleleft)$ and $(T \times T, [[\cdot,\cdot,\cdot]], \triangle)$) is a truss which is called **left (resp. right and medium) hemisemi-direct product**.

A map $\mathcal{K} : T \to T$ is a left (resp. right) averaging operator if and only if its graph

$$\mathrm{Gr}(\mathcal{K}) := \{(\mathcal{K}(x), x),\ x \in T\}$$

is a sub-truss of left (resp. right) hemisemi-direct product on $T \times T$.

Corollary

A map $\mathcal{K} : T \to T$ is a homomorphic averaging operator if and only if its graph $\mathrm{Gr}(\mathcal{K})$ is a sub-truss of left, right and medium hemisemi-direct product on $T \times T$.

Proposition

Let $\mathcal{K}$ be an averaging operator on a truss $(T, [\cdot,\cdot,\cdot])$. Then, $(A, [\cdot,\cdot,\cdot], \vdash, \dashv)$ is a di-truss, where

$$x \vdash y = \mathcal{K}(x) \cdot y \text{ and } x \dashv y = x \cdot \mathcal{K}(y), \quad \forall\, x, y \in T. \tag{74}$$

Moreover, if $\mathcal{K}$ is homomorphic then $(A, [\cdot,\cdot,\cdot], \vdash, \dashv, \perp := \cdot)$ is a tri-truss.