WARING TYPE PROBLEMS IN GROUPS AND ALGEBRAS

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- Similar questions have been considered in group theory and for algebras.
- We will speak about some of them.

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- The subgroup generated by $w(G)$ is called the *verbal subgroup*, $\langle w(G) \rangle$.
- The word w has *finite widt* N on G if $\langle w(G) \rangle = \underbrace{w(G)^{\pm 1} \cdots w(G)^{\pm 1}}_{N}$ and N is minimal satisfying the above condition.

Some results

- Ore considered the word $\omega = [x, y]$ and proved that this word has width 1 in every alternating A_n, $n \geq 5$, that is, any element of A_n is a commutator.

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- A. Shalev (2009) proved that if G is a finite simple group, big enoug, then the width of an arbitrary word ω in G is ≤ 3.

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- Finally, Liebeck, O'Brien, Shalev and Tiep proved Ore conjecture. Every element in a simple group G is a commutator.
- Nikolov and Seagal proved that if G is a m-generated finite group, then the subgroup generated by the nth-powers of elements of G is

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- Consequenlty a subgroup of finite index in a f.g. profinite group is closed

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- Let K be a commutative unital ring and $f = f(X_1, \ldots, X_m)$ a noncommutative polynomial. We can consider $f(A) = \{f(a_1, \ldots, a_m) | a_i \in A\}$ and its linear span $spanf(A)$.

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- Then f has finite width in A if there is $N \geq 1$ s.t. every element in $spanf(A)$ is a linear combination of N elements in $f(A)$. The smallest such N is called the width of f in A, $W_{f,A}$ and the *Waring constant* of A is: $W_A = supW_{f,A}$.

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- $W_{M_n} \leq 5$ for every $n \geq 2$. It is an open problem to compute W_{M_n}.

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- D. Vitas proved that if K is algebraically closed of characteristic 0, given a multilinear polynomial f of degree ≤ 3, the verbal set $f(M_n(K))$ is a vector space for every $n \geq 2$.

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- Let f be a nonconstant noncommutative polynomial with coefficients in a field K. Then an arbitrary traceless $n \times n$ matrix , n sufficiently large, can be expressed as the difference of two elements from $f(M_n(K))$: $sl_n \subseteq f(M_n) - f(M_n)$.

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- If a polynomial is not a sum of commutators, then for n large enough, every nonscalar matrix in M_n is a linear combination of two matrices from $f(M_n)$.

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- If the topology is complete, then G is a pro-p-group. If not, the pro-p-completion of G, G_p, is considered.

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- For a pro-p-group G and $w = w(x_1, \ldots, x_n) \in F$ we consider the set

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  of all values of the word w in G.
- Then $\langle \overline{w(G)} \rangle$ denotes the closed subgroup of G generated by the set $w(G)$.

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- (EZ) Let G be a pro-p group satisfying a non-trivial pro-p identity ω. If G has a dense f.g. torsion discrete group, then G is finite.

Multilinear words

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- Let $w \in F$ be a word. Suppose $w \in F_n \setminus F_{n+1}$. The word w is *multilinear* if $w = \bar{w}w'$ where
  1. $\bar{w}$ is a product of left-normed commutators of length n such that the element $\bar{w}F_{n+1}$ does not belong to $pF(L)$, with $F(L) = \oplus_{i\geq 1} F_i/F_{i+1}$;
  2. $w' \in F_{n+1}$ is a (converging) product of commutators in $x_1, \ldots, x_n$ of length $\geq n+1$ and each commutator involves all n generators $x_1, \ldots, x_n$.

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- **Th. B** Given an arbitrary word $w \in F$ there is a multilinear word $\tilde{w} \in \langle \overline{w(G)} \rangle$
- The proof imitates the linearization process in algebras.

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$$w(G, i, \alpha) = \{w(a_1, \ldots, a_{i-1}, g, a_{i+1}, \ldots, a_n) \mid g \in G\}.$$

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$$w(G, i, \alpha) = \{w(a_1, \ldots, a_{i-1}, g, a_{i+1}, \ldots, a_n) \mid g \in G\}.$$

- The element $w \in F$ is *strongly elliptic* on G if there exist finite subsets $M_i \subseteq \underbrace{G \times \cdots \times G}_{n-1}$, $1 \leq i \leq n$, and an order on $\cup_{i=1}^{n} M_i = \{\alpha_1 < \alpha_2 < \cdots < \alpha_q\}$, $\alpha_k \in M_{i_k}$, $1 \leq k \leq q$, $1 \leq i_k \leq n$, such that

$$\langle \overline{w(G)} \rangle = w(G, i_1, \alpha_1)^{\pm 1} \cdots w(G, i_q, \alpha_q)^{\pm 1}.$$

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- A result of J.P. Serre can be seen as the proof that the commutator is strongly elliptic on finitely generated pro-p-groups.
- **Th. C** If Γ is a finitely generated residually-p torsion group, then an arbitrary multilinear word is strongly elliptic on the pro-p-group $G = \Gamma_p$.

Lie Algebras

- Let $f(x_1, \ldots, x_n)$ be a multilinear element in $Lie < X >$, L a K-Lie algebra and $f(L) = \{f(a_1, \ldots, a_n) | a_i \in L\}$.

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- We say that the element f is *strongly elliptic* on L if there exist finite sets $M_i \subseteq \underbrace{L \times \cdots \times L}_{n-1}$, $1 \leq i \leq n$, such that $Span(f(L))$ is a sum of additive subgroups $f(a_1, \ldots, a_{i-1}, L, a_{i+1}, \ldots, a_n)$, where $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in M_i$, for $1 \leq i \leq n$.

Theorem [CM]

- Let L be a K- Lie algebra generated by a finite set such that an arbitrary commutator in the generators of L is ad-nilpotent (generators are considered as commutators of length 1). Let $0 \neq f(x_1, \ldots x_n) \in Lie < X >$ a multilinear element of the free Lie K-algebra. Then
  1. $I = Span(f(L))$ is an ideal of L and $|L : Span(f(L))| < \infty$,
  2. f is strongly elliptic on L.

Associative nil algebras

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- **Theorem** Let A be a finitely generated nil associative algebra. An arbitrary nonzero multilinear element from $K\langle X\rangle$ is strongly elliptic on A.

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- **Theorem** Let A be a finitely generated nil associative algebra. An arbitrary nonzero multilinear element from $K\langle X\rangle$ is strongly elliptic on A.
- In what follows A will denote a finitely generated nil associative algebra

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- (*Sketch of the Proof.*) Alahmadi and Alsulami proved that if A is a finitely generated associative algebra then $[A^{(-)}, A^{(-)}]$ is also finitely generated.

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- (*Sketch of the Proof.*) Alahmadi and Alsulami proved that if A is a finitely generated associative algebra then $[A^{(-)}, A^{(-)}]$ is also finitely generated.
- Since A is nil, all commutators in the generators of $A^{(-)}$ are ad-nilpotent.

Some needed results

- **Lemma 1** Suppose that a nonzero multilinear element f lies in $Lie < X >$. Then f is strongly elliptic on A.
- (*Sketch of the Proof.*) Alahmadi and Alsulami proved that if A is a finitely generated associative algebra then $[A^{(-)}, A^{(-)}]$ is also finitely generated.
- Since A is nil, all commutators in the generators of $A^{(-)}$ are ad-nilpotent.
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- Since A is nil, all commutators in the generators of $A^{(-)}$ are ad-nilpotent.
- The result in [CM] assures that f is strongly elliptic on $A^{(-)}$.
- Strong ellipticity in A and $A^{(-)}$ are the same thing, what completes the proof of Lemma 1

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- Let $g(x_1, \ldots, x_{d+2})$ be the multilinear Lie polynomial that is skew-symmetric in $x_1, \ldots, x_{d+1}$,

$$\sum_{\sigma \in S_{d+1}} (-1)^{\sigma} ad(x_{\sigma(1)}) \ldots ad(x_{\sigma(d+1)}) x_{d+2}$$

- Then $g = 0$ identically in all the algebras A_i, hence $g(x_1, \ldots, x_{d+2}) \in Jac(F_{\mathcal{V}} < X >)$.

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- Consequently, there exists $r \geq 1$ such that

$$\bar{g}(x_1, \ldots x_{d+3}) = ad(g(x_1, \ldots x_{d+2}))^r x_{d+3} = 0.$$

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- It remains to consider the full linearization of the polynomial $\bar{g}$.

- **Lemma 3** Let $f(x_1, \ldots, x_n) \in K < X >$ be a multilinear element. Then

$$[x_{2n+1}[f(x_1, \ldots, x_n), f(x_{n+1}, \ldots, x_{2n})]x_{2n+2}, x_{2n+3}] \in$$

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- The proof of this lemma follows some arguments due to I. Herstein (see [IH]).
- Now using the three previous lemmas we can prove the following proposition needed to prove the theorem.

- **Proposition** For an arbitrary nonzero multilinear element $f \in K < X >$ the space $Spanf(K < X >)$ contains a nonzero multilinear Lie polynomial.

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- *Proof* Let $f(x_1, \ldots, x_n)$ be a nonzero multilinear element of $K < X >$.
- Let $g(x_1, \ldots, x_{2n}) = [f(x_1, \ldots, x_n), f(x_{2n+1}, \ldots x_{2n})]$.

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- Let $g(x_1, \ldots, x_{2n}) = [f(x_1, \ldots, x_n), f(x_{2n+1}, \ldots x_{2n})]$.
- By Lemma 2 there exists a multilinear nonzero Lie polynomial $h(x_1, ..., x_m) \in K < X > g(K < X >)K < X >$.
- By Lemma 3, the Lie multilineal polynomial $[h(x_1, \ldots, x_m), x_{m+1}] \in Span(f(K < X >))$.

- If $h(x_1, \ldots, x_m)$ is a multilinear Lie polynomial and $h(x_1, \ldots, x_m) \in Span(f(K < X >))$ then

$$h(x_1, \ldots, x_m) = \sum \alpha f(v_1, \ldots, v_n), \qquad (2)$$

where $\alpha \in K$, $v_1, \ldots, v_n$ are words in $x_1, \ldots, x_m$ and each letter x_i occurs in $v_1, \ldots, v_n$ once.

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- Let $h(x_1, \ldots, x_m)$ be a nonzero multilinear Lie polynomial that belongs to $Span(f(K < X >))$.
- By the result in Lie algebras, $Span(h(A))$ has finite codimension in A.
- Hence,

$$Span(f(A)) = Fu_1 + \ldots + Fu_r + Span(h(A)),\ u_i \in f(A),\ 1 \leq i \leq r$$

- By Lemma 1 h is strongly elliptic on A, hence

$$Span(h(A)) = \sum_i h(a_{i_1}, \ldots, A, \ldots, a_{i_{m-1}})$$

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- By (2),

$$h(a_1, \ldots, A, \ldots, a_{m-1}) \subseteq \sum_j f(b_{j_1}, \ldots, A, \ldots, b_{j_{n-1}}),$$

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- This proves that the element f is strongly elliptic on A.

CONGRATULATIONS MANUEL