

Morita Theory for Quantaes

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Categorical, homological and computational methods in Algebra
A conference honouring the mathematical contribution of Manuel Ladra
Celebrating his 70th birthday
Santiago de Compostela, June 24-26, 2026

¹The author gratefully acknowledges the support by the Shota Rustaveli National Science Foundation Grant FR-23-271.

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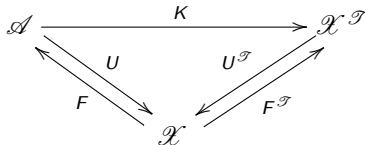
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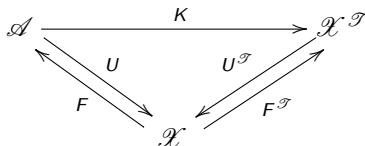
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- Given: an adjunction $\eta, \varepsilon : F \dashv U : \mathcal{A} \rightarrow \mathcal{X}$, a monad $\mathcal{T} = (T, \mu, \eta)$ on $\mathcal{X}$ and a functor $K : \mathcal{A} \rightarrow \mathcal{X}^{\mathcal{T}}$ with $U^{\mathcal{T}} K = U$.



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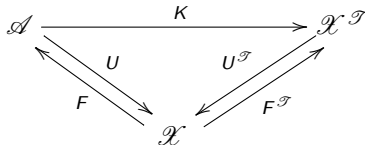
- Let $\mathcal{S}$ be the monad on $\mathcal{X}$ generated by the adjunction $F \dashv U$; and γ_K be the composite

$$F^{\mathcal{T}} \xrightarrow{F^{\mathcal{T}} \eta} F^{\mathcal{T}} U F = F^{\mathcal{T}} U^{\mathcal{T}} K F \xrightarrow{\varepsilon^{\mathcal{T}} K F} K F.$$

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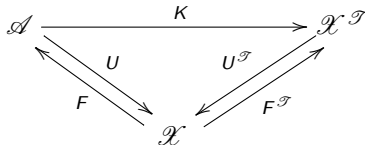
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The Comparison Theorem [B. Mesablishvili, R. Wisbauer, J. Algebra 324 (2010)464–506.]

K is an equivalence of categories if and only if the functor U is monadic and t_K is an isomorphism of monads ($= U$ is $\mathcal{T}$ -Galois).

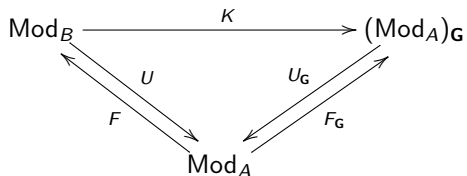
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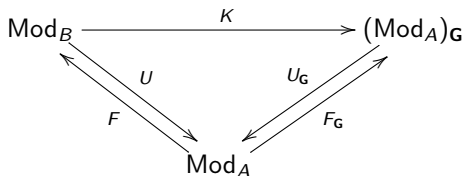
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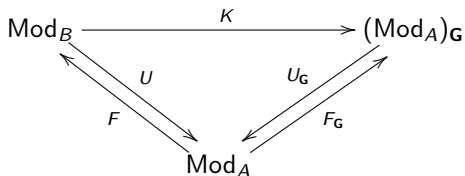


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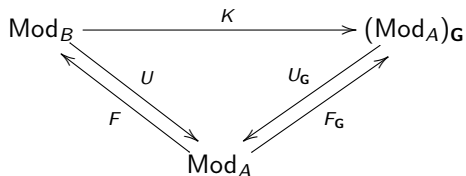


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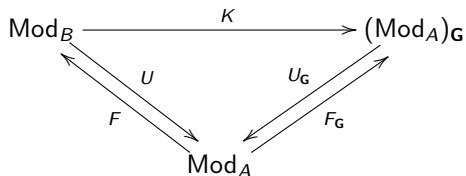


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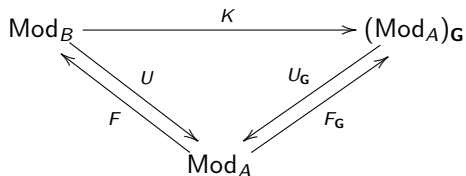


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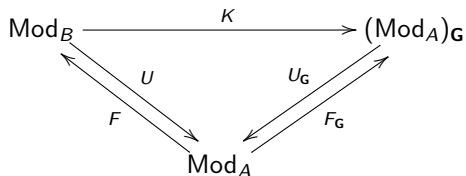
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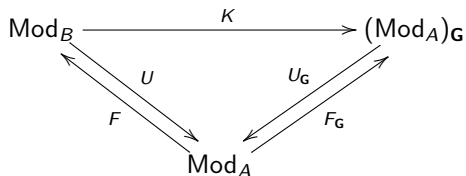
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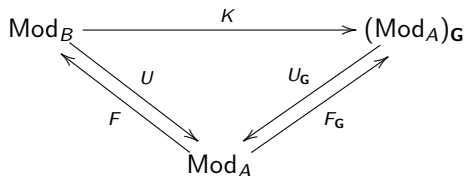
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Quantaloids

- A *quantaloid* is a locally small category enriched in the symmetric monoidal closed category $\mathcal{S}l$.
- If $\mathcal{Q}$ is a quantaloid, then one has the following commutative diagram of categories and functors:

$$\begin{array}{ccc}
 & & \mathcal{S}l \\
 & \nearrow \text{Hom}_{\mathcal{Q}}(-,-) & \downarrow \mathcal{S}l(2,-) \\
 \mathcal{Q}^{\text{op}} \times \mathcal{Q} & \xrightarrow{\mathcal{Q}(-,-)} & \text{Set},
 \end{array}$$

in which $\text{Hom}_{\mathcal{Q}}(-,-)$ is the so called *lifted hom-functor*.

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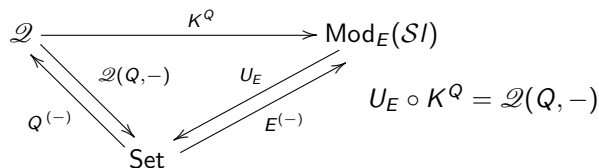
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- Let A be an arbitrary quantale. If X and Y are sets, an A -matrix a of size $X \times Y$ is a family $(a(x, y))_{(x, y) \in X \times Y}$ of elements of A (equivalently, a map $X \times Y \rightarrow A$).

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Two quantales A and B are Morita equivalent if and only if there exist a set X and a full idempotent $a \in \text{Mat}_X(A)$ such that $B \simeq a \star \text{Mat}_X(A) \star a$ as quantales.

Morita equivalence in terms of matrices

We characterize Morita equivalence for quantales via matrices and full idempotents in the following two theorems.

Theorem

Let A be a quantale. For any nonempty set X , A and $\text{Mat}_X(A)$ are Morita equivalent quantales.

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Let $\mathcal{E}$ be a Grothendieck topos, and let $\mathcal{S}l(\mathcal{E})$ denote the category of internal sup-lattices in $\mathcal{E}$. By applying our main theorem, we can deduce:

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Theorem

The category $Sl(\mathcal{E})$ of internal sup-lattices in a Grothendieck topos $\mathcal{E}$ is equivalent to the external module category $Mod_{Sub(B \times B)}(Sl)$, where B is a bound of $\mathcal{E}$ (i.e., an object of $\mathcal{E}$ whose subobjects form a generating family for $\mathcal{E}$).

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Let $\mathcal{E}$ be a Grothendieck topos, and let $SI(\mathcal{E})$ denote the category of internal sup-lattices in $\mathcal{E}$. By applying our main theorem, we can deduce:

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The category $SI(\mathcal{E})$ of internal sup-lattices in a Grothendieck topos $\mathcal{E}$ is equivalent to the external module category $Mod_{Sub(B \times B)}(SI)$, where B is a bound of $\mathcal{E}$ (i.e., an object of $\mathcal{E}$ whose subobjects form a generating family for $\mathcal{E}$).

As an immediate corollary we have:

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- A **locale** is a commutative idempotent quantale L such that $I \leq 1$ for all $I \in L$.
- For any object X of a Grothendieck topos $\mathcal{E}$, the set $\text{Sub}(X)$ is a locale.
- Since any locale L is a partially ordered set, it can be seen as a category L whose objects are the elements of L , and there is a morphism $I \rightarrow I'$ in L if and only if $I \leq I'$. Define a Grothendieck topology J on L by assigning to each element I of L (i.e. object of L) the collection $J_I = \{(I_i \leq I)_{i \in I}\}$ such that $\bigvee_{i \in I} I_i = I$.

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- Let $\text{Sh}(L)$ be the category of sheaves on the site (L, J) .

Theorem (Joyal-Tierney)

For any locale L , there is a natural equivalence

$$SI(\text{Sh}(L)) \simeq \text{Mod}_L(SI).$$

Thank you!