Morita Theory for Quantales

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The Comparison Theorem

- Given: an adjunction $\eta, \varepsilon : F \dashv U : \mathscr{A} \to \mathscr{X}$, a monad $\mathscr{T} = (T, \mu, \eta)$ on $\mathscr{X}$ and a functor $K : \mathscr{A} \to \mathscr{X}^{\mathscr{T}}$ with $U^{\mathscr{T}} K = U$.

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- Let $\mathscr{S}$ be the monad on $\mathscr{X}$ generated by the adjunction $F \dashv U$; and γ_K be the composite

$$F^{\mathscr{T}} \xrightarrow{F^{\mathscr{T}}\eta} F^{\mathscr{T}} UF = F^{\mathscr{T}} U^{\mathscr{T}} KF \xrightarrow{\varepsilon^{\mathscr{T}} KF} KF.$$

Then $t_K = U^{\mathscr{T}} \gamma_K$ is a monad morphism $\mathscr{T} \to \mathscr{S}$.

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The Comparison Theorem [B. Mesablishvili, R. Wisbauer, J. Algebra 324 (2010)464–506.]
K is an equivalence of categories if and only if the functor U is monadic and t_K is an isomorphism of monads (=U is $\mathscr{T}$-Galois).

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- The category $\mathcal{S}l$ of sup-lattices is monadic over Set via the functor $\mathcal{S}l(2, -) : \mathcal{S}l \to \mathrm{Set}$;
- (Small) coproducts are biproducts; that is, if $\{X_i\}$ is a (small) family of objects of $\mathcal{S}l$, the comparison morphism $\coprod X_i \to \prod X_i$ is an isomorphism.

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$$a * (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a * b_i) \text{ and } (\bigvee_{i \in I} a_i) * b = \bigvee_{i \in I} (a_i * b) \tag{1}$$

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A morphism $f : A \to B$ of quantales is a morphism of sup-lattices satisfying

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Morita equivalence in terms of matrices

- Let A be an arbitrary quantale. If X and Y are sets, an *A-matrix* a of size $X \times Y$ is a family $(\mathsf{a}(x,y))_{(x,y)\in X\times Y}$ of elements of A (equivalently, a map $X \times Y \to A$).
- If a and b are two A-matrices with size $X \times Y$ and $Y \times Z$, then their product $\mathsf{a} \star \mathsf{b}$ has size $X \times Z$ and is defined by

$$(\mathsf{a} \star \mathsf{b})(x,z) = \bigvee_{y\in X} \mathsf{a}(x,y)\mathsf{b}(y,z).$$

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Two quantales A and B are Morita equivalent if and only if there exist a set X and a full idempotent $a \in Mat_X(A)$ such that $B \simeq a \star Mat_X(A) \star a$ as quantales.

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- For any object X of a Grothendieck topos $\mathcal{E}$, the set $\mathrm{Sub}(X)$ is a locale.
- Since any locale L is a partially ordered set, it can be seen as a category L whose objects are the elements of L, and there is a morphism $l \to l'$ in L if and only if $l \leq l'$. Define a Grothendieck topology J on L by assigning to each element l of L (i.e. object of L) the collection $J_l = \{(l_i \leq l)_{i \in I}\}$ such that $\bigvee_{i \in I} l_i = l$.

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Theorem (Joyal-Tierney)

For any locale L, there is a natural equivalence

$$\mathcal{S}l(Sh(L)) \simeq Mod_L(\mathcal{S}l).$$

Thank you!