Relative Gorenstein dimensions under change of rings results

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Overview

1. Introduction and motivation

2. Gorenstein homological algebra

3. Relative Gorenstein homological algebra under change of rings results

Throughout this presentation:

- R will denote an associative (not necessarily commutative) ring with identity
- All R-modules are unitary R-modules.
- All R-modules, except mentioned otherwise, are left R-modules.

Regular ring

Theorem (Auslander, Buchbaum and Serre (1955-1956))

A Noetherian local commutative ring R is regular if and only if $\mathrm{gldim}(R) < \infty$.

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Every regular local commutative ring R is a unique factorization domain (UFD).

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$$\boxed{\text{Homological Algebra}} \longleftrightarrow \boxed{\text{Ring Theory}}$$

Classical homological dimension under Change of ring result

Hilbert's syzygy Theorem

Let $R[X]$ be the polynomial ring over R. Then:

$$\operatorname{gldim}(R[X]) = \operatorname{gldim}(R) + 1.$$

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Let $R[X]$ be the polynomial ring over R. Then:

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Theorem

If S is a multiplicative subset of a commutative ring R, then

$$\mathrm{gldim}(S^{-1}R) \leq \mathrm{gldim}(R).$$

Gorenstein rings

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Question (Auslander, 1967)

Which homological invariants characterize Gorenstein rings, in the same way that they characterize regular rings?

Historical overview of the Gorenstein dimension

$\diamondsuit$ 1967-1969, Auslander and Bridger

G-dimension of finitely generated modules over a commutative Noetherian ring.

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$\diamondsuit$ 1995, Enochs and Jenda
Gorenstein projective and injective modules over arbitrary modules and rings.

Gorenstein projective, injective module

Definition (Enochs and Jenda, 1995)

An R-module M is said to be Gorenstein projective (simply, G-projective), if there is an exact sequence

$$\mathbf{P}_\bullet : \quad \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{-1} \xrightarrow{f_{-1}} P_{-2} \xrightarrow{f_{-2}} \cdots$$

of projective R-module such that $\mathbf{P}_\bullet$ is $\mathrm{Hom}_R(-, Q)$ exact for every projective R-module Q and $M = \mathrm{Im} f_0$.
The Gorenstein injective module are defined similarly using injective module.

Gorenstein dimensions of module

Definition (Enochs and Jenda, 1995)

A module M is said to have G-projective dimension less than or equal to n, $\mathrm{Gpd}(M) \leq n$, if there is an exact sequence

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$$

with G_i is G-projective module for every $i \in \{0, \ldots, n\}$. If n is the least nonnegative integer for which such a sequence exists, then $\mathrm{Gpd}(M) = n$, and if there is no such n then $\mathrm{Gpd}(M) = \infty$. The Gorenstein injective dimension of M ($\mathrm{Gid}_R(M)$) is defined as the dual of the G-projective dimension of M.

Classical homological dimensions and the Gorenstein homological dimensions

For every R-module M, we have:

- $\mathrm{Gpd}_R(M) \leq \mathrm{pd}_R(M)$ with equality if $\mathrm{pd}_R(M) < +\infty$.
- $\mathrm{Gid}_R(M) \leq \mathrm{id}_R(M)$ with equality if $\mathrm{id}_R(M) < +\infty$.

Gorenstein global dimension

Theorem (Bennis and Mahdou, 2010)

For every ring R, we have:

$$\sup\{\mathrm{Gpd}_R(M),\ M \text{ is an } R-\text{module}\} = \sup\{\mathrm{Gid}_R(M),\ M \text{ is an } R-\text{module}\}$$

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This common value is called **Gorenstein global dimension** of R, and denoted by $\mathbf{Ggldim(R)}$.

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This common value is called **Gorenstein global dimension** of R, and denoted by $\mathbf{Ggldim(R)}$.

Proposition

For every ring R, we have:

$$\mathrm{Ggldim}(R) \leq \mathrm{gldim}(R)$$

With equality if $\mathrm{gldim}(R) < \infty$.

The Gorenstein dimensions are used to characterize Gorenstein rings

Theorem

For a Noetherian local commutative ring R, the following statements are equivalent:

- *R is Gorenstein (i.e., $\mathrm{id}_R(R) < \infty$).*
- *$\mathrm{Gpd}_R(M) < \infty$ for every R-module M.*
- *$\mathrm{Gid}_R(M) < \infty$ for every R-module M.*
- *$\mathrm{Ggldim}(R) < \infty$.*

Gorenstein counterpart of Hilbert's syzygy Theorem

Theorem (Bennis and Mahdou, 2009)

Let $R[X]$ be the polynomial ring over R. Then:

$$\mathrm{G}-\mathrm{gldim}(R[X]) = \mathrm{G}-\mathrm{gldim}(R) + 1.$$

New trend in homological Gorenstein algebra

$\diamondsuit$ 2006, Holm and Jørgensen
C-Gorenstein homological dimensions as a relative aspect of Gorenstein homological dimensions.

Notations

- Let C be an R-module, we use $\mathrm{Add}_R(C)$ to denote the class of all R-modules which are isomorphic to direct summands of direct sums of copies of C.
- Let U be an R-module, we use $\mathrm{Prod}_R(U)$ to denote the class of all R-modules which are isomorphic to direct summands of direct product of copies of U.

Relative Gorenstein projective, injective modules

Definition (2006, Holm and Jørgensen (2006))

An R-module M is said to be G_C-projective if there exists an exact sequence of R-modules

$$\mathbf{X} = \cdots \to P_1 \to P_0 \to A^0 \to A^1 \to \cdots$$

where the $P_i's$ are all projective, $A^i \in Add_R(C)$ for every $i \in \mathbb{N}$, $M \cong \mathrm{Im}(P_0 \to A^0)$, and such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{X}$ exact whenever $Q \in Add_R(C)$. We use $G_C P(R)$ to denote the class of all G_C-projective R-modules.

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Relative Gorenstein injective modules are defined dually (we denote by $G_CI(R)$ the class of all relative Gorenstein injective modules).

Motivation

- Let C be a semidualizing (R, S)-bimodule. Then there are two bijective correspondences:

$$\begin{tikzcd} G_C P(R) \arrow[r, shift left] & GP(R \ltimes C) \arrow[l, shift left] \end{tikzcd}$$

$$\begin{tikzcd} G_C I(R) \arrow[r, shift left] & GI(R \ltimes C) \arrow[l, shift left] \end{tikzcd}$$

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- Is the condition on C to be semidualizing necessary so that the relative homological algebra preserve their properties?
- Bennis, Garcia Rozas, and Oyonarte (2016) answered this question and defined what are called weakly Wakamatsu tilting and w-cotilting R-modules.

Relative Gorenstein dimensions of module

Definition (Holm and Jørgensen, 2005)

An R-module M is said to have G_C-projective dimension less than or equal to n, $G_C - pd(M) \leq n$, if there is an exact sequence

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$$

such that G_i is a G_C-projective R-module for every $i \in \{0, \ldots, n\}$. If n is the least nonnegative integer for which such a sequence exists, then $G_C - pd(M) = n$, and if there is no such n then $G_C - pd(M) = \infty$. The relative Gorenstein injective dimension is defined dually.

Relative Gorenstein Global dimensions

Definition (Holm and Jørgensen, 2005)

1. Given any $C \in$ *R-Mod*, the relative *Gorenstein global projective dimension* of R is :

$$\mathbf{G_C - PD}(\boldsymbol{R}) = \mathbf{sup}\{\mathbf{G_C - pd}_{\boldsymbol{R}}(\boldsymbol{M}) \mid \boldsymbol{M \text{ is an } R\text{-}module}\}$$

2. Given any $U \in$ *R-Mod*, the relative *Gorenstein global injective dimension* of R is :

$$\mathbf{G_U - ID}(\boldsymbol{R}) = \mathbf{sup}\{\mathbf{G_U - id}_{\boldsymbol{R}}(\boldsymbol{M}) \mid \boldsymbol{M \text{ is an } R\text{-}module}\}$$

Change of ring results for w-tilting module

Definition (Bennis, Garcia Rozas and Oyonarte, 2016)

An R-module C is weakly Wakamatsu tilting (w-tilting for short) if it satisfies the following two properties:

1. $\mathrm{Ext}_R^{i\geq 1}(C, C^{(I)}) = 0$ for every set I (that is, $C^{(I)} \in Add_R(C)^{\perp}$ for every set I).
2. R has an exact right $\mathrm{Add}_R(C)$-resolution $\mathbf{X} : 0 \to R \to C_0 \to C_1 \to C_2 \to \cdots$.

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Theorem (Bennis and Nassir)

Let $R \to S$ be a ring homomorphism and S is a flat right R-module. Then, if C is a w-tilting R-module with $\mathcal{P}_C - \mathrm{pd}_R(S \otimes_R C) < \infty$, then $S \otimes_R C$ is a w-tilting S-module.

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Proposition (Amzil, Bennis and Nassir, 2026)

If C is weakly Wakamatsu tilting R-module, then $C[X]$ is weakly Wakamatsu tilting $R[X]$-module.

G_C-projective dimensions under change of ring results

Theorem (Bennis and Nassir)

Let $R \to S$ be a ring homomorphism where S is a flat right R-module and let C be a w-tilting R-module with $\mathcal{P}_C - \mathrm{pd}_R(S \otimes_R C) < \infty$ and M is an R-module. Then,

$$\mathrm{G_{S\otimes_R C}} - \mathrm{pd}_S(S \otimes_R M) \leq \mathrm{G_C} - \mathrm{pd}_R(M).$$

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Proposition (Amzil, Bennis and Nassir, 2026)

Let M be an R-module. Then, we have:

$$G_{C[X]} - pd_{R[X]}(M[X]) \leq G_C - pd_R(M).$$

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Proposition (Amzil, Bennis and Nassir, 2026)

Let C be a weakly Wakamatsu tilting R-module. Then, for any R-module M we have:

$$G_C - pd_R(M) \leq G_{C[X]} - pd_{R[X]}(M[X]).$$

Relative G-projective Counterpart Hilbert's syzygy Theorem

Theorem (Amzil, Bennis and Nassir, 2026)

Let $R[X]$ be the polynomial ring over R and C be a weakly Wakamatsu tilting R-module. Then:

$$\mathrm{G_{C[X]}} - \mathrm{PD}(R[X]) = \mathrm{G_C} - \mathrm{PD}(R) + 1.$$

Change of ring results for w-cotilting module

Definition (Bennis, Garcia Rozas and Oyonarte, 2016)

An R-module U is said to be weakly cotilting (w-cotilting for short) if it satisfies the following two conditions:

1. $\mathrm{Ext}_R^{i\geq 1}(U^X, U) = 0$ for every set X.
2. There is an injective cogenerator R-module D with an exact left $\mathrm{Prod}_R(U)$-resolution
$$\cdots \to A_1 \to A_0 \to D \to 0.$$

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Theorem (Bennis and Nassir)

Let $R \to S$ be a ring homomorphism and let S be a projective R-module. Then, if U is a w-cotilting R-module, then $\mathrm{Hom}_R(S, U)$ is a w-cotilting S-module.

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Proposition (Amzil, Bennis and Nassir, 2026)

If U is a w-cotilting R-module, then $\mathrm{Hom}_R(R[X], U)$ is a w-cotilting $R[X]$-module.

G_C-injective dimensions under change of ring results

Theorem (Bennis and Nassir)

Let $R \to S$ be a ring homomorphism with S is a projective R-module, let U be a w-cotilting R-module and let M be an R-module. Then,

$$\mathrm{G}_{\mathrm{Hom_R(S,U)}} - \mathrm{id}_S(\mathrm{Hom}_R(S, M)) \leq \mathrm{G_U} - \mathrm{id}_R(M).$$

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Proposition (Amzil, Bennis and Nassir, 2025)

Let M be an R-module. Then, we have

$$G_{\mathrm{Hom}_R(R[X],U)} - id_{R[X]}(\mathrm{Hom}_R(R[X], M)) \leq G_U - id_R(M).$$

Relative G-injective Counterpart Hilbert's syzygy Theorem

Proposition (Amzil, Bennis and Nassir, 2025)

Let U be a w-cotilting R-module. Then, for any ring R

$$\mathrm{G}_{\mathrm{Hom}_R(R[X],U)}\text{-ID}(R[X]) \leq \mathrm{G}_U\text{-ID}(R) + 1.$$

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Proposition (Amzil, Bennis and Nassir, 2025)

Let U be a w-cotilting R-module and $G_U - ID(R) = 0$. Then

$$G_{\mathrm{Hom}_R(R[X],U)} - ID(R[X]) = 1.$$

References

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Thank you for your attention !