COHOMOLOGICAL RIGIDITY OF SOLVABLE LIE ALGEBRAS OF MAXIMAL RANK

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Introduction

The second adjoint cohomology group $H^2(\mathfrak{g}, \mathfrak{g})$ plays a central role in Lie theory, deformation theory [1], and in the geometry of the variety of Lie algebra laws [2]. Its importance stems from three fundamental facts:

- it controls infinitesimal deformations of Lie algebra structures;
- its vanishing implies cohomological rigidity;
- it provides a powerful invariant for distinguishing non-isomorphic Lie algebras.

The deformation theory initiated by Gerstenhaber[3] for associative algebras was later extended to Lie algebras in the foundational work of Nijenhuis and Richardson.[4]

[1] F. Grunewald, J. O'Halloran, *Deformations of Lie algebras*, J. Algebra, 1993

[2] F. Grunewald, J. O'Halloran, *Varieties of nilpotent Lie algebras of dimension less than six*, J. Algebra, 1988

[3] M. Gerstenhaber, *On the deformation of rings and algebras*, I, Ann. of Math., 1964

[4] A. Nijenhuis, R.W. Richardson, *Cohomology and deformations in graded Lie algebras*, Bull. Amer. Math. Soc., 1966

For a fixed dimension n, all Lie algebra structures on $\mathbb{C}^n$ form an algebraic variety $\mathcal{L}_n$, equipped with a natural action of GL_n. The orbits under this action, $\mathrm{Orb}(-)$, correspond to the isomorphism classes of Lie algebras. A Lie algebra is called *rigid* if its orbit is open in $\mathcal{L}_n$. To describe the variety of Lie algebras, it suffices to classify all rigid algebras (families).

A fundamental theorem of Nijenhuis and Richardson shows that cohomological rigidity implies geometric rigidity. However, the existence of rigid Lie algebras that are not cohomologically rigid demonstrates that the converse does not hold in general.[5]

[5] M. Ancochea Bermúdez, *On the rigidity of solvable Lie algebras*, Deformation Theory of Algebras and Structures and Applications, 1988

A particularly rich and structured class consists of solvable Lie algebras of maximal rank, which can be written as $\mathcal{R}_{\mathcal{T}} = \mathcal{N} \rtimes \mathcal{T}$, where $\mathcal{N}$ is a nilpotent Lie algebra of maximal rank and $\mathcal{T}$ is a maximal torus acting diagonally on $\mathcal{N}$. Thanks to a recent uniqueness result[6] on maximal solvable extensions of certain nilpotent Lie algebras, one can conclude that in most cases such extensions can be reduced to the solvable Lie algebras of the form $\mathcal{R}_{\mathcal{T}}$.

The classical work of Leger and Luks[7] established sufficient conditions for the vanishing of $H^2(\mathfrak{g}, \mathfrak{g})$ for a certain class of solvable Lie algebras. These conditions are formulated in terms of combinatorial relations among the weights arising in the toral decomposition of $\mathcal{N}$.

[6] B.A. Omirov, G.O. Solijanova, *On maximal solvable extensions of nilpotent Lie algebras*, Journal of Algebra, 2026

[7] G. Leger, E. Luks, *Cohomology theorems for Borel-like solvable Lie algebras in arbitrary characteristic*, Canad. J. Math., 1972

Over a field of characteristic zero, Whitehead's second lemma implies that

$$H^2(\mathfrak{s},\mathfrak{s}) = 0$$

for every finite-dimensional semisimple Lie algebra $\mathfrak{s}$. By contrast, for nilpotent Lie algebras, a classical result of Dixmier [8] establishes a non-vanishing property:

Theorem

Let $\mathfrak{n}$ be a finite-dimensional nilpotent Lie algebra over a field of characteristic zero. Then

$$H^2(\mathfrak{n},\mathfrak{n}) \neq 0.$$

In the solvable case, however, the situation is more flexible: $H^2(\mathfrak{g},\mathfrak{g})$ may vanish or not, depending on the structure of $\mathfrak{g}$.

In this talk, we will provide **sufficient conditions for the vanishing** of $H^2(\mathfrak{g},\mathfrak{g})$ for certain classes of solvable Lie algebras.

[8] See J. Dixmier, *Cohomologie des algèbres de Lie nilpotentes*, Acta Sci. Math., 1955.

For a given Lie algebra $\mathfrak{g}$, we define the *lower central* and *derived series* to the sequences of two-sided ideals defined recursively as follows:

$$\mathfrak{g}^1 = \mathfrak{g},\ \mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}],\quad k \geq 1;\quad \mathfrak{g}^{[1]} = \mathfrak{g},\ \mathfrak{g}^{[s+1]} = [\mathfrak{g}^{[s]}, \mathfrak{g}^{[s]}],\quad s \geq 1.$$

Definition

A Lie algebra $\mathfrak{g}$ is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $\mathfrak{g}^n = 0$ (respectively, $\mathfrak{g}^{[m]} = 0$).

Solvable Lie algebra of maximal rank.

Definition

A torus of a Lie algebra $\mathfrak{g}$ is a commutative subalgebra of $Der(\mathfrak{g})$, consisting of semisimple endomorphisms. A torus is said to be maximal if it is not strictly contained in any other torus.

In the case of algebraically closed field semisimple endomorphism means diagonalizable. Choosing an appropriate basis of a Lie algebra we can bring the mutually commuting diagonalizable derivations into a diagonal form, simultaneously.

Note that the equality $\dim \mathcal{T} = \dim(\mathcal{N}/\mathcal{N}^2)$ for a maximal torus $\mathcal{T}$ on $\mathcal{N}$ characterizes $\mathcal{N}$ as a nilpotent Lie algebra of maximal rank [9]. Accordingly, the Lie algebra $\mathcal{R}_{\mathcal{T}} = \mathcal{N} \rtimes \mathcal{T}$ is referred to as a solvable Lie algebra of maximal rank.

[9] D.J. Meng, Zhu L. Sheng, Solvable complete Lie algebras I, Commun. Algebra, 1996

Hoschild-Serre factorization theorem.

Theorem

Let $\mathcal{R}_{\mathcal{T}}$ be a solvable Lie algebra of maximal rank and let $\mathcal{M}$ be a $\mathcal{R}_{\mathcal{T}}$-module such that the action of $\mathcal{T}$ on $\mathcal{M}$ is diagonal. Then

$$H^n(\mathcal{R}_{\mathcal{T}}, \mathcal{M}) \cong \sum_{i+j=n} H^i(\mathcal{T}, \mathbb{F}) \otimes H^j(\mathcal{N}, \mathcal{M})^{\mathcal{T}}, \quad n \geq 0,$$

where

$$H^j(\mathcal{N}, \mathcal{M})^{\mathcal{T}} = \{f \in H^j(\mathcal{N}, \mathcal{M}) \mid (t \cdot f) = 0, \ t \in \mathcal{T}\}$$

is the space of $\mathcal{T}$-invariant cocycles of $\mathcal{N}$ with values in $\mathcal{M}$ and the invariance being defined by

$$(t \cdot f)(z_1, z_2, \ldots, z_b) = t \cdot f(z_1, z_2, \ldots, z_b) - \sum_{s=1}^{b} f(z_1, \ldots, [t, z_s], \ldots, z_j).$$

Remark

Taking into account that $H^i(\mathcal{T}, \mathbb{F}) = \wedge^i \mathcal{T}$ we conclude that $H^n(\mathcal{R}_{\mathcal{T}}, \mathcal{M}) = 0$ if and only if $H^j(\mathcal{N}, \mathcal{M})^{\mathcal{T}} = 0$ for all $0 \leq j \leq n$.

Weight-space decomposition of $\mathcal{N}$

Let $\mathcal{T}$ be a maximal torus on $\mathcal{N}$. A linear functional $\alpha \in \mathcal{T}^*$ is called a *weight* of $\mathcal{N}$ if the corresponding *weight space*

$$\mathcal{N}_\alpha = \{x \in \mathcal{N} \mid [t, x] = \alpha(t)\, x \text{ for all } t \in \mathcal{T}\}$$

is nonzero, i.e., $\mathcal{N}_\alpha \neq 0$. Consider the weight-space decomposition of $\mathcal{N}$ relative to $\mathcal{T}$,

$$\mathcal{N} = \bigoplus_{\alpha \in W} \mathcal{N}_\alpha,$$

where W denotes the set of all weights of $\mathcal{N}$ with respect to $\mathcal{T}$.

Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ denotes the set of primitive weights. Then every weight $\mu \in W$ admits a unique expression of the form

$$\mu = \sum_{i=1}^{n} r_i \alpha_i, \qquad r_i \in \mathbb{Z}_{\geq 0}.$$

Leger–Luks' classical results

Below we adapt the assumptions (i)–(iii) given in [10] to the case of $\mathbb{F} = \mathbb{C}$.

(i) $\dim \mathcal{T} = \dim(\mathcal{N}/\mathcal{N}^2) = n$;

(ii) For $\alpha \in W$, $\dim \mathcal{N}_\alpha = 1$ and, if $\alpha, \beta, \alpha + \beta$ are all in W, then $[\mathcal{N}_\alpha, \mathcal{N}_\beta] = \mathcal{N}_{\alpha+\beta}$;

(iii) If $\alpha, \beta, \gamma, \delta, \alpha + \gamma, \beta + \delta$ are all in W with α, β primitive $(\alpha \neq \beta)$ and with $\alpha + \gamma = \beta + \delta$, then there is some $\mu \in W$ such that $\delta = \alpha + \mu$, $\gamma = \beta + \mu$ and at least one of the following is satisfied:

Case 1. $\alpha + \beta \notin W$;
Case 2. $\alpha + \beta \in W$ but $\alpha + 2\beta \notin W$ and $\mu = \beta + \nu$ for some $\nu \in W$;
Case 3. $\alpha + \beta \in W$ but $2\alpha + \beta \notin W$ and $\mu = \alpha + \nu$ for some $\nu \in W$.

Then one of the main result of this work related to the assumptions given above states:

If $\mathcal{R}_\mathcal{T}$ satisfies the assumptions $(i) - (iii)$, then $H^2(\mathcal{R}_\mathcal{T}, \mathcal{R}_\mathcal{T}) = 0$.

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[10] G. Leger, E. Luks, Cohomology theorems for Borel-like solvable Lie algebras in arbitrary characteristic, Canad. J. Math., 1972

Since for an arbitrary solvable Lie algebra of the form $\mathcal{N} \oplus \mathcal{Q}$, where $\mathcal{Q}$ is complementary subspace to $\mathcal{N}$, we have upper bound estimation[11], $\dim \mathcal{Q} \leq \dim(\mathcal{N}/\mathcal{N}^2)$, then the assumption (i) implies that $\mathcal{R}_{\mathcal{T}}$ is a maximal extension of $\mathcal{N}$.

Throughout the talk, we consider a nilpotent Lie algebra $\mathcal{N}$ of maximal rank whose weight spaces respective to a maximal torus are all one-dimensional.

Thanks to the result in [12], $\mathcal{R}_{\mathcal{T}}$ is the unique (up to isomorphism) maximal solvable extension of $\mathcal{N}$. Consequently, without loss of generality, we may consider $\mathcal{R}_{\mathcal{T}}$ in place of any maximal solvable extension of $\mathcal{N}$.

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[11] L. Šnobl, On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras, J. Phys. A: Math. Theor. 2010

[12] B.A. Omirov, G.O. Solijanova, *On maximal solvable extensions of nilpotent Lie algebras,* Journal of Algebra, 2026

Therefore, we shall assume that an algebra $\mathcal{R}_{\mathcal{T}}$ satisfies conditions (i)–(ii). In what follows, if a weight λ satisfies property (iii), we will say that "λ satisfies the Leger-Luks' condition".

Below, we present the definition of cohomological rigidity [13] .

Definition

A Lie algebra $\mathcal{G}$ is called cohomologically rigid if $H^2(\mathcal{G}, \mathcal{G}) = 0$.

[13] A. Nijenhuis, R.W. Richardson, Deformations of Lie algebra structures, J. Math. Mech. 1967

Thanks to results of the paper [14] one can conclude the following statement:

A maximal solvable extension of a nilpotent Lie algebra of maximal rank has trivial center and it admits only inner derivations.

Due to Hoschild-Serre factorization theorem and the above statement, we conclude the following

Let $\mathcal{R}$ be a maximal solvable extension of a nilpotent Lie algebra of maximal rank. Then $H^2(\mathcal{R},\mathcal{R}) = 0$ if and only if $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$.

[14] B.A. Omirov, G.O. Solijanova, *On maximal solvable extensions of nilpotent Lie algebras,* Journal of Algebra, 2026.

Proposition

Let $\mathcal{N}$ be a nilpotent Lie algebra of maximal rank satisfying $\mathcal{N}^3 = 0$. Then $\mathcal{R}_{\mathcal{T}}$ is cohomologically rigid.

From now on, we focus on solvable Lie algebras $\mathcal{R}_{\mathcal{T}}$ with a nilradical of nilindex greater than 3.

We now present one of our principal results.

Theorem A

Let $\mathcal{R}_{\mathcal{T}}$ be a maximal solvable Lie algebra of maximal rank and let $\mathcal{M}$ be an $\mathcal{R}_{\mathcal{T}}$-module such that the action of $\mathcal{T}$ on $\mathcal{N}$ is diagonal on $\mathcal{M}$ and the weight of $\mathcal{T}$ in $\mathcal{M}$ are in W. If for each weight λ of $\mathcal{N}$ at least one of the following conditions holds true

i) Leger-Luks' conditions;

ii) $\lambda = \alpha_i + \mu$ for some $\alpha_i \in \Pi$ and $\lambda - \alpha_j \notin W$ for all $\alpha_j \in \Pi \setminus \{\alpha_i\}$;

iii) $\lambda = \alpha_i + \alpha_j$ or $\lambda = \alpha_i + (\alpha_i + \alpha_j)$ for some distinct $\alpha_i, \alpha_j \in \Pi$.

Then $H^2(\mathcal{N}, \mathcal{M})^{\mathcal{T}} = 0$.

Corollary

Let $\mathcal{R}_{\mathcal{T}}$ be a solvable Lie algebra satisfying the conditions of Theorem A. Then $\mathcal{R}_{\mathcal{T}}$ is cohomologically rigid.

Applications of Theorem A

Recall the cohomological rigidity of solvable Lie algebras of maximal rank in dimensions not exceeding 9 was established in [15]. In particular, the algebras

$$\mu_5^1,\ \mu_6^2,\ \mu_7^5,\ \mu_7^6,\ \mu_8^{22},\ \mu_8^{25},\ \mu_8^{26},\ \mu_8^{33}$$

from that classification satisfy conditions (ii) and (iii) of Theorem **??**.

Similarly, the maximal solvable extensions of the model filiform Lie algebra n_1 also satisfy the assumptions of cases (ii) and (iii), and their cohomological rigidity is established in [16].

[15] J.M. Ancochea Bermúdez, M. Goze, On the classification of rigid Lie algebras, J. Algebra., 2001

[16] M. Goze, Yu. Khakimdjanov, *Nilpotent Lie algebras*. Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, 1996

Moreover, the extensive computations required to establish the cohomological rigidity of the maximal solvable extension of the model nilpotent Lie algebra, as carried out in [17], can be considerably simplified by applying Theorem A, since this solvable algebra satisfies both conditions (ii) and (iii).

[17] J.M. Ancochea Bermúdez, R. Campoamor-Stursberg, Cohomologically rigid solvable Lie algebras with a nilradical of arbitrary characteristic sequence, Linear Algebra Appl., 2016

We now proceed with the following results, which will be useful in the study of high-degree cohomology groups.

Lemma

Let $\mathcal{R}_{\mathcal{T}}$ be a solvable Lie algebra of maximal rank and nilindex of $\mathcal{N}$ is $(s+1)$. Let $\mathcal{M}$ be an $\mathcal{R}_{\mathcal{T}}$-module on which the action of $\mathcal{T}$ on $\mathcal{N}$ is diagonal, and assume that all $\mathcal{T}$-weights occurring in $\mathcal{M}$ lie in W. Then $H^s(\mathcal{N}, \mathcal{M})^{\mathcal{T}} = 0$.

Proposition

Suppose $\mathcal{R}_{\mathcal{T}}$ satisfies the conditions of the previous lemma and $H^i(\mathcal{R}_{\mathcal{T}}, \mathcal{R}_{\mathcal{T}}) = 0$ for all $0 \leq i \leq s-1$. Then $H(\mathcal{N}, \mathcal{M})^{\mathcal{T}} = 0$.

To facilitate the analysis of higher-degree cohomology groups, we now present the following result.

Theorem B

Suppose $\mathcal{R}_{\mathcal{T}}$ be a solvable Lie algebra with nilradical $\mathcal{N}$ of maximal rank and nilindex $(s+1)$ such that $H^i(\mathcal{R}_{\mathcal{T}}, \mathcal{R}_{\mathcal{T}}) = 0$ for all $0 \leq i \leq s-1$. Then $H(\mathcal{R}_{\mathcal{T}}, \mathcal{R}_{\mathcal{T}}) = 0$.

Below we present an example that satisfies the assumptions of the above theorem.

Example

Let $\mathfrak{n}$ be the nilpotent Lie algebra with multiplication table

$$\begin{cases} [e_1, e_j] & = & e_{j+1}, & 2 \leq j \leq n_1, \\ [e_1, e_{n_1+\cdots+n_i+j}] & = & e_{n_1+\cdots+n_i+j+1}, & 2 \leq j \leq n_{i+1}, \quad 1 \leq i \leq k-1. \end{cases}$$

where $2 \leq n_i \leq 4$ for all $1 \leq i \leq k$.

This Lie algebra, for arbitrary values of the parameters n_i, was introduced in [18] as the model nilpotent Lie algebra.

[18] J.M. Ancochea Bermúdez, R. Campoamor-Stursberg, Cohomologically rigid solvable Lie algebras with a nilradical of arbitrary characteristic sequence, Linear Algebra Appl., 2016

Cohomologically rigid Lie agebras with rank two

Theorem

Let $\mathcal{R}_{\mathcal{T}}$ be a solvable Lie algebra of maximal rank whose nilradical is generated by two elements and has nilindex at most 5. Then $\mathcal{R}_{\mathcal{T}}$ is cohomologically rigid.

In the following theorem we give triviality of the second cohomology groups of $\mathcal{R}_{\mathcal{T}}$ with adjoint module.

Theorem

Let $\mathcal{R}_{\mathcal{T}}$ be a solvable extension of a nilpotent Lie algebra $\mathcal{N}$ of maximal rank two. Assume further that $3\alpha_1 + \alpha_2 \notin W$ and $\alpha_1 + 3\alpha_2 \notin W$. Then $\mathcal{R}_{\mathcal{T}}$ is cohomologically rigid.

Below we present one-parametric family of nilpotent Lie algebras of maximal rank that satisfies to assumptions of the above theorem.

Example

For each integer $n \geq 1$ and each parameter $t \in \mathbb{C}$, consider the $3(n+1)$-dimensional nilpotent Lie algebra $\mathfrak{g}_{n,t}$ with nonzero brackets given by

$$\begin{cases} [e_{(i-1)\alpha_1+i\alpha_2},\, e_{(j-i+1)\alpha_1+(j-i)\alpha_2}] = (-1)^{\delta_{i,1}}\, e_{j\alpha_1+j\alpha_2}, \\ [e_{i\alpha_1+i\alpha_2},\, e_{(j-i+1)\alpha_1+(j-i)\alpha_2}] = (-1)^{\delta_{i,j}}\, e_{(j+1)\alpha_1+j\alpha_2}, \\ [e_{i\alpha_1+i\alpha_2},\, e_{(j-i)\alpha_1+(j-i+1)\alpha_2}] = -(-1)^{\delta_{i,j}}\, e_{j\alpha_1+(j+1)\alpha_2}, \\ [e_{\alpha_1},\, e_{n\alpha_1+(n+1)\alpha_2}] = t\, e_{(n+1)\alpha_1+(n+1)\alpha_2}, \\ [e_{(i-1)\alpha_1+i\alpha_2},\, e_{(n-i+2)\alpha_1+(n-i+1)\alpha_2}] = t\, e_{(n+1)\alpha_1+(n+1)\alpha_2}, \end{cases}$$

where $\delta_{i,j}$ denotes the Kronecker delta and $1 \leq i \leq j \leq n$.

Non-cohomologically rigidity of $\mathcal{R}_{\mathcal{T}}$

Below, we present the structural configurations that lead to the non-vanishing of the second cohomology group.

Theorem C

Suppose $\mathcal{R}_{\mathcal{T}}$ be a solvable extension of a nilpotent Lie algebra $\mathcal{N}$ of maximal rank such that there exist $\alpha, \beta, \gamma \in \Pi$ satisfying one of the following conditions:

$$
\begin{array}{llllll}
i) & e_{3\alpha+2\beta} \in \mathrm{Center}(\mathcal{N}) & \textit{and} & \{3\alpha+\beta,\ 2\alpha+2\beta\} & \subseteq & W; \\
ii) & e_{4\alpha+2\beta} \in \mathrm{Center}(\mathcal{N}) & \textit{and} & \{4\alpha+\beta,\ 3\alpha+2\beta, 2\alpha+2\beta\} & \subseteq & W; \\
iii) & e_{\alpha+\beta+\gamma} \in \mathrm{Center}(\mathcal{N}) & \textit{and} & \{\alpha+\beta,\ \alpha+\gamma,\ \beta+\gamma\} & \subseteq & W.
\end{array}
$$

Then $H^2(\mathcal{R}_{\mathcal{T}}, \mathcal{R}_{\mathcal{T}}) \neq 0$.

From the proof of Theorem C, we obtain the following corollary.

Corollary

Let m, n and p denote, respectively, the number of pairs satisfying **Case (i)** and **Case (ii)** and the number of triples satisfying **Case (iii)**. Then the dimension of the second cohomology group of $\mathcal{R}_{\mathcal{T}}$ is bounded below by $m + n + p$.

For each weight space $\mathcal{N}_\alpha = \mathrm{Span}\{e_\alpha\}$ with $\alpha \in W$, we denote by $c(\alpha, \beta)$ the corresponding structure constant, defined by

$$[e_\alpha, e_\beta] = c(\alpha, \beta)\, e_{\alpha+\beta}.$$

Example

Consider the 9-dimensional nilpotent Lie algebra $\mathcal{N}_9$ of maximal rank two whose only non-zero structure constants are the following (each equal to 1):

$$\begin{cases} c(\alpha_1, \alpha_2),\ c(\alpha_1, \alpha_1 + \alpha_2),\ c(\alpha_1, \alpha_1 + 2\alpha_2),\ c(\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2), \\ c(\alpha_1, 2\alpha_1 + \alpha_2),\ c(\alpha_1, 2\alpha_1 + 2\alpha_2),\ c(\alpha_2, \alpha_1 + \alpha_2),\ c(\alpha_2, 2\alpha_1 + \alpha_2), \\ c(2\alpha_1 + 2\alpha_2, \alpha_2), \quad c(\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2). \end{cases}$$

From the above structure constants, one can see that $\mathcal{N}_9$ satisfies only condition (i) of Theorem C. Consequently, for 11-dimensional solvable Lie algebra $\mathcal{R}_\mathcal{T}$ of $\mathcal{N}$ we have $H^2(\mathcal{R}_\mathcal{T}, \mathcal{R}_\mathcal{T}) \neq 0$.

Example

Consider the 10-dimensional nilpotent Lie algebra $\mathcal{N}_{10}$ of maximal rank two whose only non-zero structure constants are the following (each equal to 1):

$$\begin{cases} c(\alpha_1, \alpha_2), & c(\alpha_1, \alpha_1+\alpha_2), & c(\alpha_1, 2\alpha_1+\alpha_2), \\ c(\alpha_1, 3\alpha_1+\alpha_2), & c(\alpha_1, 2\alpha_1+2\alpha_2), & c(\alpha_1, 3\alpha_1+2\alpha_2), \\ c(\alpha_2, 2\alpha_1+\alpha_2), & c(\alpha_2, 4\alpha_1+\alpha_2), & c(\alpha_1+\alpha_2, 2\alpha_1+\alpha_2), \\ c(\alpha_1, \alpha_1+2\alpha_2), & c(\alpha_2, \alpha_1+\alpha_2), & c(\alpha_1+\alpha_2, 3\alpha_1+\alpha_2). \end{cases} \tag{1}$$

From (1), we conclude that $\mathcal{N}_{10}$ does not satisfy condition (i) and (iii), but does satisfy condition (ii) of Theorem C. It follows that the 12-dimensional solvable Lie algebra $\mathcal{R}_{\mathcal{T}}$ constructed from $\mathcal{N}_{10}$ has non-trivial second cohomology.

Example

Consider the 7-dimensional nilpotent Lie algebra $\mathcal{N}_7$ of maximal rank three whose only non-zero structure constants are the following (each equal to 1):

$$c(\alpha_1, \alpha_2),\ c(\alpha_1, \alpha_3),\ c(\alpha_2, \alpha_3),\ c(\alpha_1 + \alpha_2, \alpha_3),\ c(\alpha_1 + \alpha_3, \alpha_2). \quad (2)$$

From (2), we conclude that $\mathcal{N}_7$ does not satisfy condition (i) and (ii), but does satisfy condition (iii) of Theorem C. It follows that the 9-dimensional solvable Lie algebra $\mathcal{R}_{\mathcal{T}}$ constructed from $\mathcal{N}_7$ has non-trivial second cohomology.

Based on examples of solvable Lie algebras with non-vanishing second cohomology groups, we propose the following conjecture:

Conjecture Let p denote the number of weights for which the difference between the maximum number of pairwise distinct triples and the maximum number of distinct pairs arising from those weights exceeds 1. Then

$$\dim H^2(\mathcal{R}_{\mathcal{T}}, \mathcal{R}_{\mathcal{T}}) \ \geq \ p.$$

Thank you for your attention!