Flows of finite-dimensional algebras, classification in a rotational flow

Utkir A. Rozikov

V.I.Romanovskiy Institute of Mathematics, Tashkent, Uzbekistan.

Abstract

A flow of algebras (FA) is a continuous-time dynamical system where each state is a finite-dimensional algebra defined by cubic matrices of structural constants satisfying a Kolmogorov-Chapman-type equation (KCE).
In this talk, we provide conditions on the multiplication of cubic matrices that ensure the KCE has a solution, focusing on algebras of cubic matrices under fixed multiplication.
Using ideas from continuous-time Markov processes, we construct a class of FAs via the matrix exponent of cubic matrices.
We also provide a time-dependent family of two-dimensional algebras and identify when algebras at different times are isomorphic.

Definitions.

Algebras of cubic matrices. Given a field F, any finite-dimensional algebra $\mathcal{A}$ can be specified up to isomorphism by giving its dimension (say m), and specifying cubic matrix $(c_{ijk})_{i,j,k=1}^{m}$, $c_{ijk} \in F$. These structure constants determine the multiplication in $\mathcal{A}$ via the following rule:

$$e_i e_j = \sum_{k=1}^{m} c_{ijk} e_k,$$

where $e_1, \ldots, e_m$ form a basis of $\mathcal{A}$.
A cubic matrix $Q = (q_{ijk})_{i,j,k=1}^{m}$ is a m^3-dimensional vector:

$$Q = \sum_{i,j,k=1}^{m} q_{ijk} E_{ijk},$$

where E_{ijk} is a m^3- cubic matrix whose (i,j,k)th entry is equal to 1 and all other entries are equal to 0.

Denote by $\mathfrak{C}$ the set of all cubic matrices over a field F. Then $\mathfrak{C}$ is an m^3-dimensional vector space over F, i.e. for any matrices $A = (a_{ijk})_{i,j,k=1}^m, B = (b_{ijk})_{i,j,k=1}^m \in \mathfrak{C}$, $\lambda \in F$, we have

$$A + B = (a_{ijk} + b_{ijk})_{i,j,k=1}^m \in \mathfrak{C}, \qquad \lambda A = (\lambda a_{ijk})_{i,j,k=1}^m \in \mathfrak{C}.$$

In general, one can fix an $m^3 \times m^3 \times m^3$- cubic matrix $\mu = \left(C_{ijk,lnr}^{uvw} \right)$ as a matrix of structural constants and give a *multiplication* of basis cubic matrices as

$$E_{ijk} *_\mu E_{lnr} = \sum_{uvw} C_{ijk,lnr}^{uvw} E_{uvw}. \tag{1}$$

Then the extension of this multiplication by bilinearity to arbitrary cubic matrices gives a general multiplication on the set $\mathfrak{C}$ and it becomes an algebra of cubic matrices (ACM), denoted by $\mathfrak{C}_\mu$.

Maksimov's multiplications

Denote $I = \{1, 2, \ldots, m\}$. We define the following multiplications:

$$E_{ijk} *_a E_{lnr} = \delta_{kl} E_{ia(j,n)r}, \tag{2}$$

where $a \colon I \times I \to I$, $(j, n) \mapsto a(j, n) \in I$, is an arbitrary associative binary operation and δ_{kl} is the Kronecker symbol.

Denote by $\mathcal{O}_m$ the set of all associative binary operations on I.

The general formula for the multiplication is the extension of (2) by bilinearity, i.e. for any two cubic matrices $A = (a_{ijk})_{i,j,k=1}^m$, $B = (b_{ijk})_{i,j,k=1}^m \in \mathfrak{C}$ the matrix $A *_a B = (c_{ijk})_{i,j,k=1}^m$ is defined by

$$c_{ijr} = \sum_{l,n\colon a(l,n)=j} \sum_k a_{ilk} b_{knr}. \tag{3}$$

Denote by $\mathfrak{C}_a \equiv \mathfrak{C}_a^m = (\mathfrak{C}, *_a)$, $a \in \mathcal{O}_m$, the ACM given by the multiplication $*_a$.

Flow of algebras

In[1], [2] we introduced and studied flow (depending on time) of finite-dimensional algebras.

Definition 1

Fix an arbitrary multiplication (not necessarily the Maksimov's multiplication) of cubic matrices, say $*_\mu$.
A family $\{\mathcal{A}^{[s,t]}:\ s,t\in\mathbb{R},\ 0\leq s\leq t\}$ of m-dimensional algebras over the field F is called a flow of algebras (FA) of type μ if the matrices $\mathcal{M}^{[s,t]}$ of structural constants (of $\mathcal{A}^{[s,t]}$) satisfy the Kolmogorov-Chapman equation (for cubic matrices):

$$\mathcal{M}^{[s,t]}=\mathcal{M}^{[s,\tau]}*_\mu\mathcal{M}^{[\tau,t]},\qquad \text{for all } 0\leq s<\tau<t. \tag{4}$$

[1] M. Ladra, U.A. Rozikov, *Jour. Algebra*, **470**, (2017) 263–288.
[2] M. Ladra, U.A. Rozikov, *Jour. Algebra*, **492**, (2017) 475–489.

Definition 2

An FA is called a (time) homogenous FA if the matrix $\mathcal{M}^{[s,t]}$ depends only on $t - s$. In this case we write $\mathcal{M}^{[t-s]}$.

Definition 3

An FA is called a discrete-time FA if the matrix $\mathcal{M}^{[s,t]}$ depends only on $t, s \in \mathbb{N}$. In this case we write $\mathcal{M}^{[n,m]}$, $n, m \in \mathbb{N}$.

To construct an FA of type μ one has to solve (4). In this talk our aim is to construct FAs, i.e. find solutions to the equation (4).

Power-associative multiplications. An algebra is *power-associative* if for each element a we have $a^n a^m = a^{n+m}$, $n, m \in \mathbb{Z}_+$.
Fix multiplication $*_\mu$ and for a cubic matrix Q denote

$$Q^{*_\mu n} = \underbrace{Q *_\mu Q *_\mu \cdots *_\mu Q}_{n \text{ times}}, \qquad n = 1, 2, \ldots .$$

Condition 1. Assume that the algebra $\mathfrak{C}_\mu$ of cubic matrices is power-associative.

Proposition 1

1. *If Condition 1 is satisfied for the multiplication $*_\mu$ then $\mathcal{M}^{[n,m]} = Q^{*_\mu (m-n)}$ generates a discrete time FA, $\mathcal{A}_1^{[n,m]}$, of type μ.*
2. *If ACM $\mathfrak{C}_\mu$ has an idempotent, i.e. there exists $\mathcal{I} \in \mathfrak{C}_\mu$ such that $\mathcal{I} *_\mu \mathcal{I} = \mathcal{I}$ then $\mathcal{M}^{[s,t]} = \mathcal{I}$, for all $0 \le s < t$, generates an FA, $\mathcal{A}_2^{[n,m]}$, of type μ.*

Remarks on time dynamics. The time dynamics of $\mathcal{A}_1^{[n,m]}$ depends on the fixed matrix Q. This FA is a time homogenous, and periodic iff the powers (with respect to multiplication μ) of the cubic matrix Q have periods, i.e. if there is $p \in \mathbb{N}$ such that $Q^{*_\mu p} = Q$; if Q is such that $\lim_{n\to\infty} Q^{*_\mu n} = \tilde{Q}$ exists then the FA has a limit algebra:

$$\tilde{\mathcal{A}}_1 = \lim_{m-n\to\infty} \mathcal{A}_1^{[n,m]}.$$

The time dynamics of $\mathcal{A}_2^{[s,t]}$ is trivial: it does not depend on time.

Exponential solutions

In the theory of continuous-time Markov chains, the matrix exponent of square matrices plays a crucial role. Here we adapt this notion of matrix exponent for cubic matrices.

An algebra is *unital* if it has an element u with $ux = x = xu$ for all x in the algebra.

Condition 2. Assume $*_\mu$ is such that the corresponding ACM, $\mathfrak{C}_\mu = (\mathfrak{C}, *_\mu)$ is unital, i.e. it has a unit matrix denoted by $\mathbb{I}$.

In the unital ACM $\mathfrak{C}_\mu$ for each cubic matrix Q we define $Q^{*_\mu 0} = \mathbb{I}$.

A *matrix exponent* $\exp_\mu(tQ)$ is defined by

$$\exp_\mu(tQ) = \mathbb{I} + \sum_{n\geq 1} \frac{(tQ)^{*_\mu n}}{n!} = \sum_{n\geq 0} \frac{(tQ)^{*_\mu n}}{n!}. \tag{5}$$

Condition 3. Assume that $\mathfrak{C}_\mu$ is a normed space with norm $\|\cdot\|$ such that for any two cubic matrices A, B, we have

$$\|A *_\mu B\| \leq \|A\|\|B\|. \tag{6}$$

Condition 4. Assume $\mathfrak{C}_\mu$ is an associative algebra.

Proposition 2

Assume Conditions 1–4 are satisfied. Let Q be a cubic matrix. Then:

(a) *The series in* (5) *converges.*

(b) *The function* $\exp_\mu(tQ)$ *is differentiable with respect to t with*

$$\frac{d}{dt}\exp_\mu(tQ) = Q *_\mu \exp_\mu(tQ) = \exp_\mu(tQ) *_\mu Q, \quad t \in \mathbb{R}.$$

(c) *The exponent matrices satisfy the semigroup property:*

$$\exp_\mu((s+t)Q) = \exp_\mu(sQ) *_\mu \exp_\mu(tQ), \quad \forall s,t \in \mathbb{R}. \tag{7}$$

(d) *The function* $\exp_\mu(tQ)$ *is the only solution to*

$$\frac{d}{dt}Y(t) = Y(t) *_\mu Q = Q *_\mu Y(t), \quad \textit{with} \quad Y(0) = \mathbb{I}. \tag{8}$$

These are Kolmogorov's forward and backward equations.

Using (7) one easily checks that $\mathcal{M}^{[s,t]} = \exp_\mu((t-s)Q)$ satisfies (4).

Summarizing, we get the following.

Theorem 3

Let μ be a multiplication such that Conditions 1–4 are satisfied. Let Q be a cubic matrix. Then the family of matrices $\{\mathcal{M}^{[s,t]} = \exp_\mu((t-s)Q)\}$, $0 \leq s < t$, generates a time-homogeneous FA of type μ, denoted by $\mathcal{A}_3^{[s,t]}$.

Time non-homogenous FAs.

The following theorem gives an example of m-dimensional time non-homogenous FA.

Let $\mathbb{I} \in \mathfrak{C}_\mu$ be a unit matrix (under Condition 2). Matrix $A^{-1} \in \mathfrak{C}_\mu$ is called inverse of an $A \in \mathfrak{C}_\mu$ if

$$A *_\mu A^{-1} = A^{-1} *_\mu A = \mathbb{I}.$$

Theorem 4

Assume $*_\mu$ such that Condition 2 and 4 are satisfied. Let $\{A^{[t]}, t \geq 0\} \subset \mathfrak{C}_\mu$ be a family of invertible (for all t) m^3-matrices. Then the matrix

$$\mathcal{M}^{[s,t]} = A^{[s]} *_\mu (A^{[t]})^{-1},$$

generates an FA of type μ, where $(A^{[t]})^{-1}$ is the inverse of $A^{[t]}$.

Two-dimensional flow of algebras

Let $m = 2$ and consider two-dimensional flow of algebras which defined by multiplication:

$$E_{ijk} \cdot E_{lnr} = \delta_{kl}\delta_{jn}E_{ijr}, \tag{9}$$

where δ_{ij}- Kronecker symbol. In this case Chapman-Kolmogorov equation (4) for 2^3-cubic matrices $\mathcal{M}^{[s,t]} = \left(c_{ijr}^{[s,t]}\right)$ has the following form:

$$c_{ijr}^{[s,t]} = c_{ij1}^{[s,\tau]}c_{1jr}^{[\tau,t]} + c_{ij2}^{[s,\tau]}c_{2jr}^{[\tau,t]}, \qquad i,j,r = 1,2.$$

The full set of solutions to the system is not known yet. But we know a very wide class of its solutions[3], [4], [5].

[3] J.M. Casas, M. Ladra, U.A. Rozikov, *Linear Algebra Appl.*, **435**(4) (2011), 852–870.

[4] A.N. Imomkulov, M.V. Velasco, *Filomat*. **34**(10) (2020), 3175-3190.

[5] B.A. Omirov, U.A. Rozikov, K.M. Tulenbayev, *Linear Multilinear Algebra*. **63**(3) (2015), 586-600.

Here we consider the following cubic matrix, which is a solution and therefore generates an FA:

$$\mathcal{M}^{[s,t]} = \left(\begin{array}{cc|cc} \cos(t-s) & \sin(t-s) & -\sin(t-s) & \cos(t-s) \\ \cos(t-s) & -\sin(t-s) & \sin(t-s) & \cos(t-s) \end{array} \right). \tag{10}$$

The algebra $A^{[s,t]}$ of the FA corresponding to the cubic matrix (10) is commutative iff $\cos(t-s) = -\sin(t-s)$, i.e. $t = s + \frac{3\pi}{4} + \pi n$, $n = 0, 1, 2, ...$ Therefore this FA is almost non-commutative (with respect to Lebesgue measure on $\mathcal{T} = \{(s,t) : 0 \leq s \leq t\}$). The FA (10) is periodic and its period is 2π.

Note that this FA is a time-homogeneous, therefore we can consider it with respect to one time parameter $\widehat{t} = t - s$. So for convenience we write $A^{[t]}$ instead of $A^{[s,t]}$.

The problem now is to classify algebras of $A^{[t]}$, i.e. at different values of time FA forms various algebras, we need to find the time-dependent conditions under which algebras are isomorphic.

Introduce the following algebras:

A_1 with structural constants matrix $\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right)$

A_0^+ with structural constants matrix $\left(\begin{array}{cc|cc} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$

A_2 with structural constants matrix $\frac{\sqrt{2}}{2}\left(\begin{array}{cc|cc} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{array}\right)$

$A_{\cos t}^{\pm}$ with $\left(\begin{array}{cc|cc} \cos t & \pm\sqrt{1-\cos^2 t} & \mp\sqrt{1-\cos^2 t} & \cos t \\ \cos t & \mp\sqrt{1-\cos^2 t} & \pm\sqrt{1-\cos^2 t} & \cos t \end{array}\right)$

Theorem 5

The following assertions hold:

1. A_0^+ *is not isomorphic to* A_1.
2. $A_{\cos t}^+$ *is isomorphic to* $A_{-\cos t}^-$ *for any* $\cos t \in (-1; 1) \setminus \{0\}$.
3. $A_{\cos t_1}^+$ *is not isomorphic to* $A_{\cos t_2}^+$ *for any* $\cos t_1, \cos t_2 \in (0; 1), \cos t_1 \neq \cos t_2$.
4. $A_{\cos t_1}^-$ *is not isomorphic to* $A_{\cos t_2}^-$ *for any* $\cos t_1, \cos t_2 \in (0; 1), \cos t_1 \neq \cos t_2$.
5. $A_{\cos t_1}^+$ *is not isomorphic to* $A_{\cos t_2}^-$ *for any* $\cos t_1, \cos t_2 \in (0; 1)$.

The following theorem[6] gives a time-depending classification FA $A^{[t]}$.

Theorem 6

$$A^{[t]} \cong \begin{cases} A_1, & \textit{if} \quad t \in \{\pi k : k \in I\} \\ A_0^+, & \textit{if} \quad t \in \{\frac{\pi}{2} + \pi k : k \in I\} \\ A_2, & \textit{if} \quad t \in \{\frac{3\pi}{4} + \pi k : k \in I\} \\ A_{\cos t}^+, & \textit{if} \quad t \in \bigcup_{k\in I}(\pi k; \frac{\pi}{2} + \pi k) \\ A_{\cos t}^-, & \textit{if} \quad t \in \bigcup_{k\in I}((\frac{\pi}{2} + \pi k; \pi + \pi k) \setminus \{\frac{3\pi}{4} + \pi k\}) \end{cases}$$

where $I = \{0, 1, 2, ...\}$ is set of nonnegative integers.

[6] U.A. Rozikov, M.V.Velasco, B.A. Narkuziyev, *Linear and Multilinear Algebra*, 2025.

We can better see the essence of the theorem using the figure below

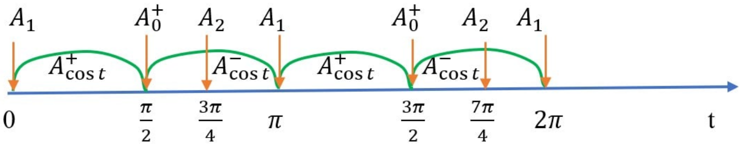

Figure: The partition of the time set $\{(0,t) : 0 \leq t\}$ corresponding to the classification of algebras in the FA $A^{[t]}$ with the matrix (10).

THANK YOU!