Associative representations of evolution algebras of dimension two

Iván Ruiz Campos. Joint work with Y. Cabrera, D. Gonçalves, M. I. Gonçalves, D. Martín and C. Martín

Universidad de Málaga. Manolofest

June 25, 2026


ENJOY
life


VI CONGRESO
DE JÓVENES
INVESTIGADORES
DE LA RSME
LEÓN 6-10
FEBRERO 2023


GIOTTO
ROBERCOLOR

Contents

A brief motivation

Consider the associative unital algebra

$$U = K[x,y]/(x^4 - x, y^4 - y, xy)$$

with involution $*\colon U \to U$ induced by $\bar{x}^* = \bar{y}^2$ and $\bar{y}^* = \bar{x}^2$.
We can define a new product $p\colon U \times U \to U$ such that $p(a,b) = a^* b^*$. If $A = K\bar{x} \oplus K\bar{y}$, then:

- $p(\bar{x},\bar{x}) = \bar{x}^*\bar{x}^* = \bar{y}^4 = \bar{y}$,
- $p(\bar{y},\bar{y}) = \bar{y}^*\bar{y}^* = \bar{x}^4 = \bar{x}$,
- $p(\bar{x},\bar{y}) = \bar{x}^*\bar{y}^* = \bar{y}^2\bar{x}^2 = 0$,

(A,p) is an evolution algebra contained in U.

A brief motivation

We can find something similar in Lie algebras:

Consider $\mathfrak{g}$ a Lie algebra, then $\mathfrak{g}$ is a subalgebra of $U(\mathfrak{g})^-$, where $U(\mathfrak{g})$ is its universal enveloping algebra.

- We gain an associative algebra.
- We have to "twist" the product.
- We obtain a "representation" $\rho\colon \mathfrak{g} \to U(\mathfrak{g})^-$ which is faithful.

$U(\mathfrak{g})$ is a Hopf algebra relative to the coalgebra operations

- $\Delta\colon U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $\Delta(x) = x \otimes 1 + 1 \otimes x$.
- $S\colon U(\mathfrak{g}) \to U(\mathfrak{g})$, $S(x) = -x$.
- $\varepsilon\colon U(\mathfrak{g}) \to K$, $\varepsilon(x) = 0$.

Evolution algebras

J. P. Tian and P. Vojtechovsky. **Mathematical concepts of evolution algebras in nonMendelian genetics.** *Quasigroups Related Systems* 14, 1 (2006), 111–122.

Definition

We say that a K-algebra A is an evolution algebra if there exists a basis $\mathcal{B} = \{e_i\}_{i \in \Lambda}$ such that $e_i e_j = 0$ if $i \neq j$. We say that this basis is *natural*.

- *It is generally non-associative.* If $e_i^2 = e_j$ and $e_j^2 \neq 0$, then $(e_i e_i)e_j = e_j^2 \neq 0$ and $e_i(e_i e_j) = 0$.
- *Commutative*: $e_i e_j = \delta_{i,j} e_i^2 = e_j e_i$.

The product is defined as

$$e_i^2 = \sum_{k \in \Lambda} \omega_{i,k} e_k$$

with a finite number of scalars $\omega_{i,j} \neq 0$.

Structure Matrix

If A is an evolution algebras with natural basis $\{e_i\}_{i\in\Lambda}$ and $e_i^2 = \sum_{j\in\Lambda} \omega_{i,j} e_j$, then we define the structure matrix of A as:

$$M_A = (\omega_{i,j})_{i,j\in\Lambda}.$$

We say that the evolution algebra is perfect if $A^2 = A$. In the finite-dimensional case, this is equivalent to $\det M_A \neq 0$.

Theorem M.L, A.P'25

Let A be a perfect evolution algebra over any field K. Then, every subalgebra of A admits a natural basis (is an evolution algebra).

Graph of an evolution algebra

A. Elduque and A. Labra **Evolution algebras and graphs.** *Journal of Algebra and Graphs* Vol. 14, No. 07, 1550103 (2015).

Given a K-evolution algebra A with basis $\mathcal{B} = \{e_i\}_{i\in\Lambda}$, we construct the associated directed graph E_A.

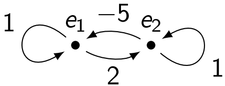
$$E_A^0 = \{v_i : i \in \Lambda\}, E_A^1 = \{f_{i,j} : \omega_{i,j} \neq 0\}$$

where $s(f_{i,j}) = v_i$ and $r(f_{i,j}) = v_j$.

Example:

Let A be the $\mathbb{R}$-evolution algebra of dimension 2 given by the basis $\mathcal{B} = \{e_1, e_2\}$ such that

$$e_1^2 = e_1 + 2e_2, e_2^2 = -5e_1 + e_2.$$



Evolution algebras of dimension 2

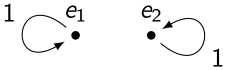
- M. I. Cardoso Gonçalves, D. Gonçalves, D. Martín Barquero, C. Martín González and M. Siles Molina. **Squares and associative representations of two-dimensional evolution algebras.** *Journal of Algebra and Its Applications* 20, 06 (2021), 2150090.

They give a full classification of all the families of isomorphism classes of evolution algebras of dimension 2:
$A_0, A_1, A_{2,\alpha}, A_{3,\alpha}, A_{4,\alpha}, A_{5,\alpha,\beta}, A_5, A_6, A_7, A_{8,\alpha}$

$$A_0 \equiv \left\{ \begin{array}{lcl} e_1^2 & = & 0 \\ e_2^2 & = & 0 \end{array} \right.$$

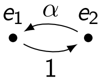
Perfect evolution algebras of dimension 2

$$A_1 \equiv \begin{cases} e_1^2 & = & e_1 \\ e_2^2 & = & e_2 \end{cases}$$



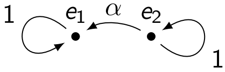
$$A_{2,\alpha} \equiv \begin{cases} e_1^2 & = & e_2 \\ e_2^2 & = & \alpha e_1 \end{cases}$$

$$\alpha \neq 0$$



$$A_{3,\alpha} \equiv \begin{cases} e_1^2 & = & e_1 \\ e_2^2 & = & \alpha e_1 + e_2 \end{cases}$$

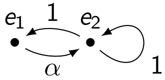
$$\alpha \neq 0$$



Perfect evolution algebras of dimension 2

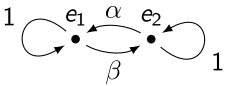
$$A_{4,\alpha} \equiv \begin{cases} e_1^2 &= \alpha e_2 \\ e_2^2 &= e_1 + e_2 \end{cases}$$

$$\alpha \neq 0$$



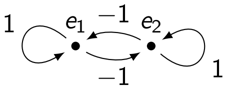
$$A_{5,\alpha,\beta} \equiv \begin{cases} e_1^2 &= e_1 + \beta e_2 \\ e_2^2 &= \alpha e_1 + e_2 \end{cases}$$

$$\alpha\beta \notin \{0,1\}$$

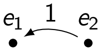


Non-perfect evolution algebras of dimension 2

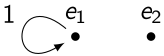
$$A_5 \equiv \begin{cases} e_1^2 = e_1 - e_2 \\ e_2^2 = -e_1 + e_2 \end{cases}$$



$$A_6 \equiv \begin{cases} e_1^2 = 0 \\ e_2^2 = e_1 \end{cases}$$

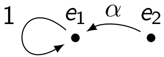


$$A_7 \equiv \begin{cases} e_1^2 = e_1 \\ e_2^2 = 0 \end{cases}$$



$$A_{8,\alpha} \equiv \begin{cases} e_1^2 = e_1 \\ e_2^2 = \alpha e_1 \end{cases}$$

$$\alpha \neq 0$$



Affine group schemes

William C. Waterhouse. **Introduction to affine group schemes**, vol. 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979.

Definition

An affine group scheme is a functor $\mathcal{F}\colon \mathrm{alg}_K \to \mathrm{Groups}$ such that $U \circ \mathcal{F}\colon \mathrm{alg}_K \to \mathrm{Groups} \to \mathrm{Sets}$ is representable.

In particular, there is an algebra $A \in \mathrm{alg}_K$ such that for every $R \in \mathrm{alg}_K$ there is a bijection

$$\mathcal{F}(R) \cong \hom_K(A, R).$$

Affine group schemes

Consider $\mathcal{F}$ an affine group scheme such that $\mathcal{F}(R)$ is a collection of touples $(r_1, \ldots, r_n) \in R^n$ which verify the equations $p_i(r_1, \ldots, r_n) = 0$ for $i = 1, \ldots, m$.

Under the previous conditions, the algebra that represents the functor $\mathcal{F}$ is

$$A = K[x_1, x_2, \ldots, x_n]/I$$

where I is the ideal generated by $p_i(x_1, \ldots, x_n)$ for $i = 1, \ldots, m$.

Examples:

- $\mu_n(R) = \{r \in R^\times : r^n = 1\}$, $A = K[x]/(x^n - 1) = KC_n$.
- $\mathrm{GL}_n(R) = \{M \in \mathcal{M}_n(R) \colon \det(M) \in R^\times\}$,
  $A = K[x_{11}, x_{12}, \ldots, x_{nn}, \det(x_{11}, x_{12}, \ldots, x_{nn})^{-1}]$.
- $G_m(R) = R^\times$, $A = K[x, x^{-1}]$.

Hopf algebra associated to an affine group scheme

If $\mathcal{F}$ is an affine group scheme, then there are morphisms

$$\Delta \colon A \to A \otimes A$$

$$S \colon A \to A$$

$$\varepsilon \colon A \to K$$

with A its representing algebra such that:

The morphisms verify:

$$\begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "Id\otimes\Delta"'] \\ A \otimes A \arrow[u, "\Delta\otimes Id"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}, \quad \begin{tikzcd} A & A \otimes A \arrow[l, "(S.Id)"] \\ K \arrow[u] & A \arrow[l, "\varepsilon"] \arrow[u, "\Delta"'] \end{tikzcd}, \quad \begin{tikzcd} K \otimes A & A \otimes A \arrow[l, "\varepsilon\otimes Id"] \\ A \arrow[u, phantom, "\wr\|"] & A \arrow[l, phantom, "="] \arrow[u, "\Delta"'] \end{tikzcd},$$

That is, A is a Hopf algebra.

Scheme of automorphisms of an evolution algebra

If R is an associative unital K-algebra and A is an evolution K-algebra, with natural basis $\{e_i\}_{i\in\Lambda}$, then $A_R := A \otimes_K R \cong \bigoplus_{i\in\Lambda} Re_i$ is an evolution R-algebra.

$$\begin{array}{rccc} \boldsymbol{aut}(A)\colon & \mathrm{alg}_K & \to & \mathrm{Groups} \\ & R & \mapsto & \boldsymbol{aut}(A)(R) := \mathrm{aut}_R(A_R) \end{array}$$

If $\varphi\colon R \to S$ is a morphism of K-algebras, then $\boldsymbol{aut}(A)(\varphi)\colon \mathrm{aut}_R(A_R) \to \mathrm{aut}_S(A_S)$ is a morphism such that for a $T \in \boldsymbol{aut}_R(A_R)$ we have that

$$\begin{array}{rccc} \boldsymbol{aut}(A)(\varphi)(T)\colon & A_S & \to & A_S \\ & a\otimes 1_S & \mapsto & (Id_A \otimes \varphi)(T(a\otimes 1_R)) \end{array}$$

Scheme of automorphisms of an evolution algebra

Alberto Elduque, Alicia Labra. **Evolution algebras, automorphisms and graphs**, *Linear Multilinear Algebra 69* 2 (2021),331-342.

Theorem

If A is a perfect finite dimensional evolution algebra, then the sequence

$$1 \to \mathrm{Diag}(E_A) \xrightarrow{\mathrm{i}} \boldsymbol{aut}(A) \xrightarrow{\rho} \mathrm{aut}(E_A)$$

is exact.

We use this theorem, or consequences of this theorem, for $A_1, A_{3,\alpha}, A_{4,\alpha}, A_{5,\alpha,\beta}$.
However, for the Non-perfect case we have to appeal to the classical computation.

Hopf algebra associated to an evolution algebra

In some cases the characteristic of the field K plays a key role. Let us consider the algebra A_5. We have that:

$$\mathrm{aut}_R(A_{5R}) \cong \left\{ \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix} : a \in R, 2a-1 \in R^\times \right\}$$

- If $\mathrm{char}(K) \neq 2$, then $\boldsymbol{aut}(A_5) \cong \mathbf{G_m}$ and $\mathcal{H}_{A_5} = K[x, x^{-1}]$

$$\begin{array}{rccc} \Delta\colon & \mathcal{H}_{A_5} & \to & \mathcal{H}_{A_5} \otimes \mathcal{H}_{A_5} \\ & x & \mapsto & x \otimes x \end{array}$$

$$\begin{array}{rccc} S\colon & \mathcal{H}_{A_5} & \to & \mathcal{H}_{A_5} \\ & x & \mapsto & x^{-1} \end{array}$$

$$\begin{array}{rccc} \varepsilon\colon & \mathcal{H}_{A_5} & \to & K \\ & x & \mapsto & 1 \end{array}$$

Hopf algebra associated to an evolution algebra

In some cases the characteristic of the field K plays a key role. Let us consider the algebra A_5. We have that:

$$\mathrm{aut}_R(A_{5R}) \cong \left\{ \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix} : a \in R, 2a-1 \in R^\times \right\}$$

- If $\mathrm{char}(K) = 2$, then $\boldsymbol{aut}(A_5) \cong \mathbf{G_a}$ and $\mathcal{H}_{A_5} = K[x]$

$$\begin{array}{cccc} \Delta\colon & \mathcal{H}_{A_5} & \longrightarrow & \mathcal{H}_{A_5} \otimes \mathcal{H}_{A_5} \\ & x & \longmapsto & x \otimes 1 + 1 \otimes x \end{array}$$

$$\begin{array}{cccc} S\colon & \mathcal{H}_{A_5} & \longrightarrow & \mathcal{H}_{A_5} \\ & x & \longmapsto & -x \end{array}$$

$$\begin{array}{cccc} \varepsilon\colon & \mathcal{H}_{A_5} & \longrightarrow & K \\ & x & \longmapsto & 0 \end{array}$$

Hopf algebra associated to an evolution algebra

A	$\boldsymbol{aut}(A)$	$\mathcal{H}_{\mathbf{A}}$
A_1	$\mathbf{C_2}$	K^2
$A_{2,\alpha}$	$\boldsymbol{\mu_3} \rtimes \mathbf{C_2}$	$KC_3 \otimes K^2$
$A_{3,\alpha}$	1	K
$A_{4,\alpha}$	1	K
$A_{5,\alpha,\beta}, \alpha \neq \beta$	1	K
$A_{5,\alpha,\alpha}$	$\mathbf{C_2}$	K^2

Table: Hopf algebras associated to the perfect evolution algebras of dimension 2.

Hopf algebra associated to an evolution algebra

$\mathbf{A}$	$\boldsymbol{aut}(A)$	$\mathcal{H}_{\mathbf{A}}$
$A_5, \mathrm{char}(K) = 2$	$\mathbf{G_a}$	$K[x]$
$A_5, \mathrm{char}(K) \neq 2$	$\mathbf{G_m}$	$K[x, x^{-1}]$
A_6	$\mathbf{G_m} \times \mathbf{G_a}$	$K[x, y, y^{-1}]$
A_7	$\mathbf{G_m}$	$K[x, x^{-1}]$
$A_{8,\alpha}, \mathrm{char}(K) \neq 2$	$\mathbf{C_2}$	K^2
$A_{8,\alpha}, \mathrm{char}(K) = 2$	1	$K[x]/(x^2)$

Table: Hopf algebras associated to the perfect evolution algebras of dimension 2.

Universal associative unital representation

Alexandr I. Kornev and Ivan P. Shestakov. **On associative representations of non-associative algebras.** *Journal of Algebra and Its Applications*. 17, 3 (2018), 1850051, 12.

Consider the polynomial algebra $K[x, y, x^*, y^*]$ with $x \mapsto x^*$, $y \mapsto y^*$ an involution. Take $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in K$ and

$$p(a, b) := \lambda_0 ab + \lambda_1 ab^* + \lambda_2 a^* b + \lambda_3 a^* b^*.$$

Let A be a two-dimensional evolution algebra with natural basis $\{e_1, e_2\}$ such that $e_1^2 = \omega_{1,1} e_1 + \omega_{1,2} e_2$ and $e_2^2 = \omega_{2,1} e_1 + \omega_{2,2} e_2$.

Goal

Build a $*$-ideal I, such that there is an homomorphism of evolution algebras $\rho\colon A \to U_p$ with $U_p := K[x, y, x^*, y^*]/I$ and $\rho(ab) = p(\rho(a), \rho(b))$.

Universal associative unital representation

I is the $*$-ideal generated by

$$\begin{gathered} p(x,x) - \omega_{1,1}x - \omega_{1,2}y \\ p(y,y) - \omega_{2,1}x - \omega_{2,2}y \\ p(x,y) \\ p(y,x) \end{gathered}$$

Definition

The morphism $\rho\colon A \to U_p$ given by $\rho(e_1) = \bar{x}$ and $\rho(e_2) = \bar{y}$ is the *associative unital p-representation of A*. The algebra U_p is the *(faithful) universal unital p-algebra.*

It is universal in the sense that if there is another associative unital p-representation $\sigma\colon A \to V$ with involution, then there is a morphism $\theta\colon U_p \to V$ of unital algebras with involution such that $\sigma = \theta \circ \rho$.

Universal associative representation

Instead of $K[x, y, x^*, y^*]$, if we make use of $\langle x, y, x^*, y^* \rangle$ (free associative commutative algebra) and repeat the previous construction, we have:

$$r\colon A \to \mathcal{T}_p$$

with $\mathcal{T}_p = \langle x, y, x^*, y^* \rangle / I$.

In this case we talk about the tight universal associative p-representation of A. And $\mathcal{T}_p$ is the *(faithful) tight universal p-algebra.*

Faithful universal associative representations

Strategy:
Consider an evolution algebra of dimension 2 given by one of the known families.

- Take the polynomial $p(a,b) = \lambda_0 ab + \lambda_1 ab^* + \lambda_2 a^* b + \lambda_3 a^* b^*$.
  - Find $(\lambda_i) \in K^4$ such that $\{\bar{x}, \bar{y}\}$ is linearly independent in U_p.
  - Explore all the possible values of $(\lambda_i) \in K^4$ and see that $\{\bar{x}, \bar{y}\}$ is linearly dependent.

Lemma

Consider A a two-dimensional evolution algebra. If $\rho \colon A \to U_p$ is a faithful p-representation such that $x^2 - (x^*)^2, y^2 - (y^*)^2 \in I$, then A is isomorphic to A_1, A_6 or A_7, that is, A is associative.

If $\rho \colon A \to U_p$ is faithful with $x - x^*, y - y^* \in I$, then A is isomorphic to A_1, A_6 or A_7, that is, A is associative.

Associative representations

A	**Universal Representation**	$\mathbf{p(a, b)}$
A_1	Faithful	ab
$A_{2,\alpha}$	Faithful	$a^* b^*$
$A_{3,\alpha}$	Not faithful	
$A_{4,\alpha}$	Not faithful	
$A_{5,\alpha,\beta}, \alpha \neq \beta$	Not faithful	
$A_{5,\alpha,\alpha}$	Faithful	$ab + \alpha a^* b^*$

Table: Existence or not existence of a faithful universal associative representation of a perfect evolution algebra of dimension 2.

Associative representations

A	**Univ. Reps.**	$\mathbf{p(a,b)}$
$A_5, \mathrm{char}(K) = 2$	Faithful	$ab + a^*b^*$
$A_5, \mathrm{char}(K) \neq 2$	Faithful	$-2ab + 2a^*b^*$
A_6	Faithful	ab
A_7	Faithful	ab
$A_{8,\alpha}, \mathrm{char}(K) \neq 2$	Faithful	$k_\alpha(ab + a^*b^*) + h_\alpha(ab^* + a^*b)$
$A_{8,\alpha}, \mathrm{char}(K) = 2$	Not faithful	

Table: Existence or not existence of a faithful universal associative representation of a Non-perfect evolution algebra of dimension 2.

Tight associative representations and Hopf algebras

We can summarize all the information that we know in the following theorem.

Theorem

For a 2-dimensional evolution algebra A over K, we have that $\mathrm{aut}(A) = \{1\}$ if and only if there are no faithful associative universal p-algebras for A. If $\mathcal{H}$ is the representing Hopf algebra of $\boldsymbol{aut}(A)$, then there is a faithful associative and unital p-representation for A if and only if one of the following hold:

- $\mathrm{char}(K) \neq 2$ and $\mathcal{H} \neq K$,
- $\mathrm{char}(K) = 2$, $\mathcal{H} \neq K[x]/(x^2)$ and $\mathcal{H} \neq K$.

Furthermore, if A is perfect and has a faithful tight p-algebra, then $\mathcal{T}_p \cong \mathcal{H}$.

(Tight) associative representations and Hopf algebras

$\boldsymbol{A}$	$\boldsymbol{aut}(A)$	$\boldsymbol{\mathcal{H}}$	**Univ. repr.**	$\boldsymbol{\mathcal{T}_p}$
A_1	C_2	K^2	Faithful	$\cong \mathcal{H}$
$A_{2,\alpha}$	$\mu_3 \rtimes C_2$	$KC_3 \otimes K^2$	Faithful	$\cong \mathcal{H}$
$A_{3,\alpha}$	1	K	Not faithful	—
$A_{4,\alpha}$	1	K	Not faithful	—
$A_{5,\alpha,\beta}$, $(\alpha \neq \beta)$	1	K	Not faithful	—
$A_{5,\alpha,\alpha}$	C_2	K^2	Faithful	$\cong \mathcal{H}$

Table: Hopf algebras and universal representations of 2-dimensional perfect evolution algebras.

Tight associative representations and Hopf algebras

A	$aut(A)$	$\mathcal{H}$	**Univ. repr.**	$\mathcal{T}_p$
A_5, char $= 2$	$\mathbf{G_a}$	$K[x]$	Faithful	
A_5, char $\neq 2$	$\mathbf{G_m}$	$K[x, x^{-1}]$	Faithful	
A_6	$\mathbf{G_m} \times \mathbf{G_a}$	$K[x, y, y^{-1}]$	Faithful	
A_7	$\mathbf{G_m}$	$K[x, x^{-1}]$	Faithful	
$A_{8,\alpha}$, char $\neq 2$	C_2	K^2	Faithful	
$A_{8,\alpha}$, char $= 2$	1	$\frac{K[x]}{(x^2)}$	Not faithful	$-$

Table: Hopf algebras and universal representations of non-perfect evolution algebras.

Bibliography

[1] Yolanda Cabrera Casado, Maria I. Cardoso Gonçalves, Daniel Gonçalves, Dolores Martín Barquero, Cándido Martín González, and Iván Ruiz Campos.
Hopf algebras and associative representations of two-dimensional evolution algebras.
Bulletin of the Brazilian Mathematical Society, New Series, 56(1):8, 2025.

[2] Maria I. Cardoso Goncalves, Daniel Gonçalves, Dolores Martín Barquero, Cándido Martín González, and Mercedes Siles Molina.
Squares and associative representations of two-dimensional evolution algebras.
Journal of Algebra and Its Applications, 20(06):2150090, 2021.

Thank you

The authors are supported by the Spanish Ministry of Science and Innovation within the projects PID2019-104236GB-I00 and PID2023-152673NB-I00 and by the Junta de Andalucía within the project FQM-336.

UNIVERSIDAD
DE MÁLAGA