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Tangent Lie algebras of automorphism groups of free algebras

(Joint work with U.U.Umirbaev)

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1. Introduction

We introduce the notion of tangent Lie algebras for certain automorphism groups of free algebras, defined as subalgebras of the Lie algebra of derivations. We show that, for many classical varieties of algebras, the tangent Lie algebra is contained in the Lie algebra of derivations with constant divergence.

We also introduce the concepts of approximately tame and absolutely wild automorphisms of free algebras in arbitrary varieties and employ tangent Lie algebras to investigate their properties.

It occurs that nearly all known examples of wild automorphisms of free algebras are absolutely wild, with the notable exceptions of the Nagata and Anick automorphisms.

We show that the Bergman automorphism of free matrix algebras of order two is absolutely wild.

Furthermore, we prove that free algebras in any variety of polynilpotent Lie algebras – except for the abelian and metabelian varieties – also possess absolutely wild automorphisms.

Let $\mathfrak{M}$ be an arbitrary variety of algebras over a filed K of characteristic zero and let $A = K_{\mathfrak{M}}\langle x_1, x_2, \ldots, x_n\rangle$ be the free algebra of $\mathfrak{M}$ with a free set of generators $X = \{x_1, x_2, \ldots, x_n\}$. Let $\mathrm{Aut}(A)$ be the group of all automorphisms of A.

Denote by $\phi = (f_1, f_2, \ldots, f_n)$ the automorphism ϕ of A such that $\phi(x_i) = f_i$, $1 \leq i \leq n$. An automorphism

$$(x_1, \ldots, x_{i-1}, \alpha x_i + f, x_{i+1}, \ldots, x_n),$$

where $0 \neq \alpha \in K$, $f \in K_{\mathfrak{M}}\langle X \setminus \{x_i\}\rangle$, is called *elementary*.

The subgroup $\mathrm{TAut}(A)$ of $\mathrm{Aut}(A)$ generated by all elementary automorphisms is called the *tame automorphism group*, and the elements of this subgroup are called *tame automorphisms* of A.

Nontame automorphisms of A are called *wild*.

The well known Jung–van der Kulk Theorem says that all automorphisms of the polynomial algebra $K[x, y]$ in two variables x, y over a field K are tame.

Similar results hold for free associative algebras (Czerniakiewicz, Makar Limanov) and for free Poisson algebras (in characteristic zero) (Makar Limanov, Umirbaev, I.Sh.).

Moreover, the automorphism groups of polynomial algebras, free associative algebras, and free Poisson algebras in two variables (over a field of characteristic zero) are isomorphic.

The automorphism groups of commutative and associative algebras generated by three elements are much more complicated. The well-known Nagata automorphism

$$(x + 2y\alpha + z\alpha^2, y + z\alpha, z), \ \alpha = zx - y^2,$$

of the polynomial algebra $K[x, y, z]$ over a field K of characteristic 0 is proven to be wild (I.Sh., U.Umirbaev)].

The Anick automorphism

$$(x + z\beta, y + \beta z, z), \ \beta = xz - zy,$$

of the free associative algebra $K\langle x, y, z\rangle$ over a field K of characteristic 0 is also proven to be wild (Umirbaev).

Recently I.Sh. and Z.Zhang constructed an analogue of the Anick automorphism for free Poisson algebras in three variables:

$$(x + z\gamma, y + \{z, \gamma\}, z), \ \gamma = \{x, z\} - yz,$$

which is wild but its polynomial replica is tame.

In 1964 P. Cohn proved that all automorphisms of finitely generated free Lie algebras over a field are tame.

Later this result was extended by J.Lewin to free algebras of Nielsen-Schreier varieties. Recall that a variety of universal algebras is called Nielsen-Schreier, if any subalgebra of a free algebra of this variety is free, i.e., an analog of the classical Nielsen-Schreier theorem is true.

The varieties of all non-associative algebras (A.Kurosh), commutative and anti-commutative algebras, Lie algebras (A.Shirshov) over a field are Nielsen-Schreier.

It was recently shown by Dotsenko and Umirbaev that the varieties of pre-Lie (also known as left-symmetric) algebras and Lie-admissible algebras over a field of characteristic zero are Nielsen–Schreier.

In particular, every automorphism of a free left-symmetric and a free Lie-admissable algebra of finite rank is tame.

An automorphism ϕ is called *approximately tame* if it can be approximated by a sequence of tame automorphisms $\{\psi_k\}_{k\geq 0}$ with respect to the *formal power series topology*. An automorphism ϕ is called *absolutely wild* if it is not approximately tame.

In 1983 Anick proved that every automorphism of the polynomial algebra $K[x_1, x_2, \ldots, x_n]$ over a field K of characteristic zero is approximately tame.

In fact, a similar result was announced by Shafarevich in 1966,1981, 1995. The topology considered by Shafarevich was different. He considered automorphism groups as examples of so called Ind-groups or ∞-dimensional algebraic groups (an inductive limit of finite dimensional algebraic varieties).

Anick directly considered the power series topology and applied it for endomorphisms too.

They both considered their results as a weak generalization of the above mentioned Jung–van der Kulk Theorem.

Unfortunately, in 2018 J.P. Furter and H.P. Kraft have found a mistake in the proof of Shafarevich. The counter-example was based on the above mentioned result on non-timeness of the Nagata automorphism.

This means that the Nagata automorphism is wild but approximately tame in the power series topology. Whether it is true for Shafarevich's Ind-groups topology, remains an open question.

An analogue of the Anick result for free associative algebras and free Poisson algebras is still unknown.

In particular, we don't know weather the above mentioned wild Anick automorphism for free associative algebra and its analogue for Poisson algebras are approximately tame or not.

We will give the formal definition of approximately tame automorphisms a bit later. Now we will define the tangent Lie algebras for subgroups of $Aut\,A$.

2 Tangent Lie algebras

Consider again the free algebra A of a variety $\mathfrak{M}$ with a free set of generators $X_n = \{x_1, x_2, \ldots, x_n\}$. If $\mathfrak{M}$ is a unitary variety of algebras then we suppose that A has the identity element 1.

Consider the natural grading

$$A = A_0 \oplus A_1 \oplus \ldots \oplus A_k \oplus \ldots$$

with respect to the standard function deg, that is, A_k is the linear span of (non-associative) monomials of degree k. A non-zero element $f \in A_k$ is called homogeneous of degree k. We have $A_0 = K \cdot 1$ if A is unital, and $A_0 = 0$ otherwise.

Let $L = \mathrm{Der}(A)$ be the Lie algebra of all derivations of A. For any n-tuple $F = (f_1, \ldots, f_n) \in A^n$ there exists a unique derivation D of A such that $D(x_i) = f_i$ for all $1 \leq i \leq n$. Denote this derivation by

$$D = f_1\partial_1 + \ldots + f_n\partial_n = D_F.$$

We say that D is homogeneous of degree i if $f_1, \ldots, f_n \in A_{i+1}$. Let L_i be the space of all homogeneous derivations of degree i. Then

$$L = L_{-1} \oplus L_0 \oplus L_1 \oplus \ldots \oplus L_k \oplus \ldots,$$

is a grading of L, that is, $[L_i, L_j] \subseteq L_{i+j}$. We have

$$L_{-1} = \begin{cases} K\partial_1 + \cdots + K\partial_n, & \text{if } 1 \in A, \\ 0, & \text{otherwise,} \end{cases}$$

and

$L_0 \cong \mathfrak{gl}(n)$, with $e_{ij} = x_i\partial_j$, $1 \leq i, j \leq n$.

We are going to study some subalgebras of L containing L_0. The following proposition describes some important properties of such subalgebras. Recall that every automorphism of A acts by conjugation on the algebra of derivations L.

Proposition. *Let R be a subalgebra of L containing L_0. Then*

(a) *R is homogeneous with respect to each variable x_i. In particular, R is homogeneous with respect to the degree function* deg.

(b) *R is invariant under the action of the group of linear automorphisms $GL_n(K)$.*

Let $\mathsf{IA}(k) \subseteq \mathsf{Aut}(A)$ be the set of all automorphisms of A that induces the identity automorphism on the factor-algebra $A/(A_{k+1} + A_{k+2} + \ldots)$.

Consider the descending central series

$\mathsf{IA}(A) = \mathsf{IA}(1) \supseteq \mathsf{IA}(2) \supseteq \ldots \supseteq \mathsf{IA}(k) \supseteq .$

Let $\phi \in \mathsf{IA}(i) \setminus \mathsf{IA}(i+1)$ for some $i \geq 1$. Then

$$\phi = (x_1 + f_1 + F_1, \ldots, x_n + f_n + F_n),$$

where $f_j \in A_{i+1}, F_j \in A_{i+2} + A_{i+3} + \ldots$ for all $1 \leq j \leq n$. The derivation

$$T(\phi) = f_1\partial_1 + \ldots + f_n\partial_n \in L_i$$

will be called *the tangent to the automorphism ϕ with respect to power series topology.*

Note that $T(\phi) \neq 0$. Set also $T(\mathrm{Id}) = 0$, where Id is the identity automorphism.

Set

$$\mathsf{Gr}_n = \begin{cases} \mathsf{Aff}_n(K) & \text{if } 1 \in A, \\ \mathsf{GL}_n(K) & \text{otherwise.} \end{cases}$$

Let $H \subseteq \mathsf{Aut}\, A$ be a subgroup such that

$$\mathsf{Gr}_n \subseteq H \subseteq \mathsf{Aut}\, A.$$

Set $H_0 = H$ and $H_i = H \cap \mathsf{IA}(i)$ for all $i \geq 1$. Then

$$H = H_0 \supseteq H_1 \supseteq H_2 \supseteq \ldots.$$

Denote

$$V_i = V_i(H) = \{0\} \cup \{T(\phi) | \phi \in H_i \setminus H_{i+1}\}$$

for all $i \geq 1$. Consider now

$$T(H) = L_{-1} \oplus L_0 \oplus V_1 \oplus V_2 \oplus \ldots \oplus V_k \oplus \ldots.$$

Recall that $L_{-1} = 0$ if A is not unital, that is, in this case

$$T(H) = L_0 \oplus V_1 \oplus V_2 \oplus \ldots \oplus V_k \oplus \ldots.$$

Theorem 1. *$T(H)$ is a graded Lie subalgebra of the Lie algebra $\mathrm{Der}(A)$.*

We call $T(H)$ the *tangent algebra* of H with respect to the power series topology.

Notice that

$$\mathrm{Aut}K[x,y] \cong \mathrm{Aut}K\langle x,y\rangle$$

but

$$T(\mathrm{Aut}K[x,y]) \ncong T(\mathrm{Aut}K\langle x,y\rangle).$$

The power series topologies of $\mathrm{Aut}K[x,y]$ and $\mathrm{Aut}K\langle x,y\rangle$ differ.

The algebra $T(H)$ is especially useful for establishing its connection to the study of automorphisms.

3. Universal enveloping algebras, Jacobian matrices, and divergence

Let B be an arbitrary algebra of $\mathfrak{M}$. The universal enveloping algebra $U(B) = U_{\mathfrak{M}}(B)$ is the associative algebra with identity generated by all universal operators of left multiplication l_x and right multiplication r_x for all $x \in B$. Every B-bimodule V in the variety of algebras $\mathfrak{M}$ can be regarded as a left U-module with respect to the action

$$l_b m = bm, r_b m = mb,\ b \in B, m \in M.$$

Conversely, every left U-module can be considered as a B-bimodule in the variety $\mathfrak{M}$.

In other words, there is an isomorphism of categories

$$B_{\mathfrak{M}}\text{-}bimod \cong U(B)_{Ass}\text{-}mod.$$

If $\mathfrak{M}$ is a unital variety and B is an algebra with identity 1, then $l_1 = r_1 = \mathrm{Id} = 1$.

We give a short explanation of the structure of $U = U(A)$ when $A = K_{\mathfrak{M}}\langle x_1, \ldots, x_n\rangle$ is a free algebra. Consider A as the subalgebra of the free algebra

$$B = K_{\mathfrak{M}}\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$$

generated by $x_1, \ldots, x_n$. Denote by Ω_A the set of all homogeneous elements of B of degree 1 with respect to the set of variables $y_1, \ldots, y_n$. For any $a \in A$ the operators

$$L_a, R_a : \Omega_A \to \Omega_A,$$
$$L_a(w) = aw,\ \ R_a(w) = wa, w \in \Omega_A,$$

are well defined. The algebra generated by all $L_a, R_a,$ where $a \in A$, is isomorphic to the algebra $U = U(A).$ Notice that

$$\Omega_A = Uy_1 \oplus \ldots \oplus Uy_n$$

is a free U-module with free generators $y_1, \ldots, y_n$.

The linear map

$$D : A \to \Omega_A$$

defined by $D(x_i) = y_i$ for all i and such that

$$D(ab) = aD(b) + D(a)b = L_a D(b) + R_b D(a)$$

for all $a, b \in A$, is called the *universal derivation* of A and Ω_A is called the *universal differential module* of A. Notice that over a field of charateristic zero $D(a) = 0$ if and only if $a \in K$.

For every $a \in A$ there exist unique elements $u_1, u_2, \ldots, u_n \in U$ such that

$$D(a) = u_1 y_1 + u_2 y_2 + \ldots + u_n y_n.$$

The elements $u_i = \frac{\partial a}{\partial x_i}$ are called *the Fox derivatives* of $a \in A$. Set also

$$\delta(a) = (\frac{\partial a}{\partial x_1}, \ldots, \frac{\partial a}{\partial x_n}).$$

Let $Y = (y_1, \ldots, y_n)^t$ where t is the transpose. Then

$$D(a) = \delta(a) Y.$$

For any $f, g_1, \ldots, g_n \in A$ we get the chain rule

$$
\begin{aligned}
D(f(g_1, \ldots, g_n)) &= \frac{\partial f}{\partial x_1}(g_1, \ldots, g_n)D(g_1) + \cdots \\
&+ \frac{\partial f}{\partial x_n}(g_1, \ldots, g_n)D(g_n).
\end{aligned}
$$

Let $\phi = (f_1, f_2, \ldots, f_n)$ be an arbitrary endomorphism of A. Denote by

$$J(\phi) = [\partial_j(f_i)]_{1 \leq i,j \leq n}$$

its Jacobian matrix. We have

$$\begin{bmatrix} D(f_1) \\ \cdot \\ \cdot \\ \cdot \\ D(f_n) \end{bmatrix} = J(\phi)Y.$$

If $\psi = (g_1, \ldots, g_n)$ is another endomorphism then

$$\phi \circ \psi = (g_1(f_1, \ldots, f_n), \ldots, g_n(f_1, \ldots, f_n)).$$

The chain rule immediately implies

$$J(\phi \circ \psi) = \phi(J(\psi))J(\phi).$$

This gives that $J(\phi)$ is invertible over U if ϕ is an automorphism.

An analogue of the Jacobian Conjecture for the free algebra A can be formulated as follows: *if $J(\phi)$ is invertible over U, then ϕ is an automorphism of A.*

This conjecture is true for

A.Yagzhev (1980), U.Umirbaev (1994,1996):
free nonassociative algebras, free commutative and anticommutative algebras;

W.Dicks and J.Lewin (1982), Schofield (1985):
free associative algebras

Reutenauer (1992),Shpilrain (1993),Umirbaev (1993), Zolotyh and Mikhalev (1995):
free Lie algebras and superalgebras.

Let $D = D_F$ be a derivation of A for some $F = (f_1, \ldots, f_n)$. The *divergence of D*, denoted by $\mathrm{div}(D)$, is the image of

$$\frac{\delta f_1}{\delta x_1} + \frac{\delta f_2}{\delta x_2} + \ldots + \frac{\delta f_n}{\delta x_n} = \mathrm{Tr}(J(D)) = \mathrm{Tr}(J(F))$$

in the quotient space

$$U/([U, U] + R),$$

where $R = \mathrm{Rad}(U)$ is the Jacobson radical of $U = U(A)$.

Every derivation D of A can be uniquely extended to a derivation D^* of U by $D^*(l_a) = l_{D(a)}, D^*(r_a) = r_{D(a)},\ a \in A$.

U.Umirbaev (2016): Let $D_1, D_2 \in \mathrm{Der}(A)$. Then

$$\mathrm{div}([D_1, D_2]) = D_1^*(\mathrm{div}(D_2)) - D_2^*(\mathrm{div}(D_1))$$

This implies that the space

$$\mathsf{SDer}(A) = S_{-1} \oplus S_0 \oplus S_1 \oplus \ldots \oplus S_k \oplus \ldots$$

of all derivations of A with zero divergence forms a subalgebra of $L = \mathsf{Der}(A)$. Notice that $S_{-1} = L_{-1}$ and $S_0 = \mathsf{sl}_n(K)$ is the subalgebra of traceless matrices of $L_0 = \mathsf{gl}_n(K)$.

This algebra will be called the *special Lie algebra of derivations* of A as in the case of polynomial algebras.

Also, the space

$$\widetilde{\mathsf{SDer}}(A) = S_{-1} \oplus L_0 \oplus S_1 \oplus \ldots \oplus S_k \oplus \ldots$$

of all derivations of A with constant divergence also is a subalgebra of L.

If A doesn't have the identity element then $S_{-1} = L_{-1} = 0$ and we have

$$\begin{aligned} \mathsf{SDer}(A) &= S_0 \oplus S_1 \oplus \ldots \oplus S_k \oplus \ldots, \\ \widetilde{\mathsf{SDer}}(A) &= L_0 \oplus S_1 \oplus \ldots \oplus S_k \oplus \ldots \end{aligned}$$

Theorem 2. *Let $\mathfrak{M}$ be one of the following varieties of algebras:*

(1) A Nielsen-Schreier variety of algebras;

(2) The variety of all associative algebras;

(3) the variety of all associative and commutative algebras;

(4) the variety of all metabelian Lie algebras.

If H is a subgroup of $\mathrm{Aut}(A)$ containing Gr_n, then the tangent algebra $T(H)$ is a subalgebra of the algebra $\widetilde{\mathrm{SDer}}(A)$ of derivations of A with constant divergence.

Problem. Is it always true that $T(\mathrm{Aut}(A))$ is a subalgebra of $\widetilde{\mathrm{SDer}}(A)$?

4. Approximately tame automorphisms and density

Consider again the descending central series

$$\mathrm{IA}(A) = \mathrm{IA}(1) \supseteq \mathrm{IA}(2) \supseteq \cdots \supseteq \mathrm{IA}(k) \supseteq \cdots$$

An automorphism $\phi \in \mathrm{Aut}(A)$ is called *approximately tame* if there exists a sequence of tame automorphisms $\{\psi_k\}_{k\geq 0}$ such that $\phi\psi_k^{-1} \in \mathrm{IA}(k)$.

An automorphism ϕ is called *absolutely wild* if it is not approximately tame.

It is well known that if $n \geq 3$ and K is a field of the characteristic 0 then

$$\mathrm{TAut}(K[X_n] = \mathrm{gr}\langle \mathrm{Aff}_n(K) \cup \{\epsilon\}\rangle,$$

where ϵ is the quadratic automorphism

$$\epsilon = (x_1 + x_2^2, x_2, \ldots, x_n).$$

Anick proved that the same subgroup is dense in the group $\mathrm{Aut}(K[X_n])$. Consequently, every automorphism of $K[X_n]$ is approximately tame.

Notice that $\mathrm{Aut}(K[x_1, x_2])$ is not even finitely generated modulo $\mathrm{Aff}_2(K)$.

Let H and N be subgroups of $\mathsf{Aut}(A)$ such that $\mathsf{Gr}_n \subseteq H \subseteq N$. We say that H *is dense in* N with respect to the power series topology if for any $i \geq 1$ and for any $\psi \in N \cap \mathsf{IA}(i)$ there exists $\phi \in H$ such that $\phi^{-1}\psi \in N \cap \mathsf{IA}(i+1)$.

Lemma. *Let* $\mathsf{Gr}_n \subseteq H \subseteq N$ *be subgroups of* $\mathsf{Aut}(A)$*. If* $T(H) = T(N)$ *then* H *is dense in* N*.*

Theorem 3. *If the tangent algebra* $T(\mathsf{Aut}(A))$ *is generated modulo* $L_{-1}+L_0$ *by all derivations of the form* $f\delta_1$*, where* $f \in K_{\mathfrak{M}}\langle x_2, \ldots, x_n\rangle$ *homogeneous of degree* ≥ 2*, then every automorphism of* A *is approximately tame.*

This theorem raises the question of determining the generators of $T(\mathsf{Aut}(A))$. By Theorem 2, $T(\mathsf{Aut}(A))$ is a subalgebra of $\widetilde{\mathsf{SDer}}(A)$ for certain varieties of algebras.

Corollary (Anick). *The subgroup* H *of* $\mathsf{Aut}(K[X_n])$ *generated by all affine automorphisms* $\mathsf{Aff}_n(K)$ *and by the quadratic automorphism* ϵ *is dense in* $\mathsf{Aut}(K[X_n])$*. In particular, the group of all tame automorphisms is dense in the group of all automorphisms.*

One of the classical and well studied varieties of algebras is the variety of metabelian Lie algebras. Let L_n be a free Lie algebra of rank n in the variables $x_1, \ldots, x_n$ over a field K. Then

$$M_n = L_n/L_n'' = L_n/[[L_n, L_n], [L_n, L_n]]$$

is the free metabelian Lie algebra of rank n in the variables $y_i = x_i + L_n''$, where $1 \leq i \leq n$.

Obviously, any nontrivial exponential automorphism of M_2 is wild since every tame automorphism in this case is linear. For the same reason they are absolutely wild.

V. Drensky (1992) proved that the exponential automorphism

$$\exp(\mathrm{ad}[y_1, y_2]) =$$

$$(y_1 + [[y_1, y_2], y_1], y_2 + [[y_1, y_2], y_2], y_3 + [[y_1, y_2], y_3])$$

of M_3 is wild. In fact, his proof shows that this automorphism is absolutely wild.

More examples of non-tame automorphisms of M_3 were constructed by Roman'kov (2008) and Nauryzbaev (2009).

In 1993 Bryant and Drensky proved that every automorphism of M_n for $n \geq 4$ is approximately tame, that is, the group of tame automorphisms is dense in $\mathrm{Aut}(M_n)$.

Below we present the direct formulations of Bryant - Drensky's results expressed in the language of tangent algebras.

Theorem. *Let M_n be a free metabelian Lie algebra over a field K of characteristic zero in the variables $y_1, \ldots, y_n$. Then the following statements are true:*

(a) If $n \geq 4$ then the algebra $\widetilde{\mathsf{SDer}}(M_n)$ is generated modulo L_0 by the derivation $[y_2, y_3]\delta_1$.

(b) The algebra $\widetilde{\mathsf{SDer}}(M_3)$ is generated modulo L_0 by the derivations $[y_2, y_3]\delta_1$ and $\mathsf{ad}([y_1, y_2])$.

Corollary (Bryant-Drensky (1993),Kofinas-Papistas (2016)).

(a) If $n \geq 4$ *then the group of automorphisms generated by all linear automorphisms* $GL_n(K)$ *and by the quadratic automorphism* $\tau = (y_1 + [y_2, y_3], y_2, \ldots, y_n)$ *is dense in* $\mathrm{Aut}(M_n)$*. In particular, the group of all tame automorphisms is dense in the group of all automorphisms.*

(b) If $n = 3$ *then the group of automorphisms generated by all linear automorphisms* $GL_3(K)$*,* τ*, and* $\exp(\mathrm{ad}[y_1, y_2])$ *is dense in* $\mathrm{Aut}(M_3)$*.*

U.Umirbaev (2024) has shown that for $n \geq 4$ over a field of characteristic different from 3, the automorphism group Aut(M_n) is generated by all linear automorphisms $GL_n(K)$, the quadratic automorphism τ, and the cubic Chein automorphism

$$(y_1 + [[y_2, y_3], y_1], y_2, \ldots, y_n).$$

This is a close analogue of the well-known Bachmuth-Mochizuki-Roman'kov Theorem which states that *every automorphism of a free metabelian group of rank ≥ 4 is tame.*

Problem. *Is the Chein automorphism tame?*

Absolutely wild automorphisms and their detection.

Obviously, every absolutely wild automorphism is wild, but not vice versa. For example, the Nagata automorphism is wild but not absolutely wild.

At the moment we don't know if the Anick automorphism is approximately tame or not. But almost all known to us other examples of wild automorphisms are absolutely wild.

Here we demonstrate a method establishing of absolutely wild automorphisms, which is a clarification of methods used by V. Drensky (1992), Papistas (1993), and Bahturin and Shpilrain (1995).

First, extend some notations for endomorphisms. Let $E = \mathrm{End}(A)$ be the monoid of all endomorphisms of A. Let $\mathrm{IE}(i) = \mathrm{IE}(A, i)$ be the submonoid of all endomorphisms of A that induces the identity automorphism on the factor-algebra

$$A/(A_{i+1} + A_{i+2} + \ldots).$$

Let $\phi \in \mathrm{IE}(i) \setminus \mathrm{IE}(i+1)$ for some $i \geq 1$. Then

$$\phi = (x_1 + f_1 + F_1, \ldots, x_n + f_n + F_n),$$

where $f_j \in A_{i+1}, F_i \in A_{i+2} + A_{i+3} + \ldots$ for all $1 \leq j \leq n$. Set

$$T(\phi) = f_1\partial_1 + \ldots + f_n\partial_n \in L_i$$

as in the case automorphisms. Set also $T(\mathrm{id}) = 0$.

Theorem 4. *Let $\mathfrak{M}$ be one of the following varieties of algebras:*

(1) A Nielsen-Schreier variety of algebras;
(2) The variety of all associative algebras;
(3) the variety of all associative and commutative algebras;
(4) the variety of all metabelian Lie algebras.
Let $\mathfrak{N}$ be a subvariety of $\mathfrak{M}$ and let $I \subseteq A$ be the ideal of all identities of $\mathfrak{N}$ in $\mathfrak{M}$ in n variables. Suppose that an endomorphism $\epsilon \in \mathsf{IE}_i(A) \setminus \mathsf{IE}_{i+1}(A)$ for some $i \geq 1$ induces an automorphism ϕ of $B = A/I$ and the ideal I does not contain elements of degree $\leq i+1$. If $\mathsf{div}(T(\epsilon)) \neq 0$ then ϕ is absolutely wild.

Let $\mathfrak{N}_c$ be the variety of all nilpotent Lie algebras of class $\leq c+1$, where $c \geq 1$, and let $\mathfrak{A}$ be the variety of all abelian Lie algebras. We have $\mathfrak{N}_1 = \mathfrak{A}$. A variety of Lie algebras of the form $\mathfrak{N}_{c_1}\mathfrak{N}_{c_2}\ldots\mathfrak{N}_{c_k}$ is called polynilpotent. In paticular, $\mathfrak{A}^2$ is the variety of all metabelian Lie algebras.

Drensky(1992), Papistas (1993), Bahturin and Shpilrain (1995) proved that free algebras of rank $n \geq 2$ in any polynilpotent variety $\mathfrak{M}$ of Lie algebras that is not $\mathfrak{A}$ and $\mathfrak{A}^2$ have wild automorphisms.

Using Theorem 4 we give a unified proof of their results.

Theorem.(Drensky(1992), Papistas (1993), Bahturin and Shpilrain (1995)) *Let $\mathfrak{M}$ be a polynilpotent variety of Lie algebras that is not $\mathfrak{A}$ and $\mathfrak{A}^2$. Then every free algebra of $\mathfrak{M}$ of rank $n \geq 2$ has absolutely wild automorphisms.*

Finally, we prove that the Bergman wild automorphism in the variety $Var(M_2(K))$ is absolutely wild.

G. Bergman proved that the endomorphism

$$\beta = (x_1 + [x_1, x_2]^2, x_2)$$

of $A = K\langle x_1, x_2\rangle$ induces a wild automorphism of the free algebra of rank two of the variety of algebras $\mathrm{Var}(M_2(K))$ generated by the matrix algebra $M_2(K)$.

Proposition. *The endomorphism β of $A = K\langle x_1, x_2\rangle$ induces an absolutely wild automorphism of the free algebra of rank two of $\mathrm{Var}(M_2(K))$.*

¡Muchas gracias!

¡Feliz Cumpleaños, Manolo!