Nonassociative Methods in Pure Matrix Algebras

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Categorical, Homological and Computational Methods in Algebra
June 23–27, 2026

Simultaneous conjugation action

$$\mathrm{GL}_n(\mathbb{C}) \curvearrowright \underbrace{\mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \times \cdots \times \mathcal{M}_n(\mathbb{C})}_{d} = \mathcal{M}_n^d, \quad T \cdot (X_1, \ldots, X_d) = (TX_1T^{-1}, \ldots, TX_dT^{-1}).$$

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L. Le Bruyn, *Orbits of matrix tuples*, Sèmin. Congr., vol. 2, Soc. Math. France, Paris (1997).

The classification of matrix tuples up to similarity has been deemed a "***hopeless problem***".
Instead, an outline of a procedure which can be seen as an approximation is given.

The first approximation to the orbit space of this action is the GIT quotient.

- **Wild problem.** $\times$ Classifying orbits $\mathcal{M}_n^d / \mathrm{GL}_n$ is intractable for $n \geq 2$.
- **Tractable.** ✓ Describe the algebra of invariants $\mathbb{C}[\mathcal{M}_n^d]^{\mathrm{GL}_n}$.

First and second fundamental theorems (Sibirskii 1968 / Razmyslov 1974 / Procesi 1976)

- **Generators:** $\mathrm{Tr}(X_{i_1} X_{i_2} \cdots X_{i_k})$ – traces of words of length $\leq n^2$.
- **Relations:** consequences of the Cayley–Hamilton theorem.

Explicit generators and relations

Find minimal generators and *all* defining relations explicitly.

Known results for small matrix sizes

$n = 2$ (Dubnov, 1940s)

$$\mathbb{C}[\mathcal{M}_2 \times \mathcal{M}_2]^{\mathrm{GL}_2} \cong \mathbb{C}\big[\mathrm{Tr}(X), \mathrm{Tr}(Y), \mathrm{Tr}(X^2), \mathrm{Tr}(XY), \mathrm{Tr}(Y^2)\big].$$

How are $\det(X)$, $\mathrm{Tr}(X^3)$, $\mathrm{Tr}(X^2Y)$ or $\mathrm{Tr}(XYXY)$ generated through these five generators?

Application of Cayley-Hamilton Theorem

$$\boxed{X^2 - \mathrm{Tr}(X)\cdot X + \det(X)\cdot I_2 = O_2}$$

$$\text{Apply trace:}\quad \mathrm{Tr}(X^2) - \mathrm{Tr}(X)\cdot \mathrm{Tr}(X) + 2\det(X)\cdot I_2 = 0$$

$$\det(X) = \frac{1}{2}\left(\mathrm{Tr}^2(X) - \mathrm{Tr}(X^2)\right)$$

$$\text{Multiply by Y:}\quad X^2Y - \mathrm{Tr}(X)\cdot XY + \det(X)\cdot Y = O_2$$

$$\text{Apply Trace:}\quad \mathrm{Tr}(X^2Y) = \mathrm{Tr}(X)\,\mathrm{Tr}(XY) - \det(X)\,\mathrm{Tr}(Y)$$

$$\text{C-H for } XY\text{:}\quad (XY)^2 - \mathrm{Tr}(XY)\cdot XY + \det(XY)\cdot I_2 = O_2$$

$$\text{Apply Trace:}\quad \mathrm{Tr}(XYXY) = \mathrm{Tr}(XY)^2 - 2\det(X)\det(Y)$$

Generators of $\mathbb{C}[M_3 \times M_3]^{\mathrm{GL}_3}$

A minimal system of eleven generators for $\mathbb{C}[M_3 \times M_3]^{\mathrm{GL}_3}$ were found[a]:

$$\begin{aligned}
&\mathrm{Tr}(X), \mathrm{Tr}(Y)\\
&\mathrm{Tr}(X^2), \mathrm{Tr}(XY), \mathrm{Tr}(Y^2),\\
&\mathrm{Tr}(X^3), \mathrm{Tr}(X^2Y), \mathrm{Tr}(XY^2), \mathrm{Tr}(Y^3),\\
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The first ten traces form a **homogeneous system of parameters (hsop)** and $\mathbb{C}[M_3 \times M_3]^{\mathrm{GL}_3}$ is Cohen-Macaulay:

$$\mathbb{C}[M_3 \times M_3]^{\mathrm{GL}_3} \cong \mathbb{C}[\textbf{hsop}] \oplus \mathrm{Tr}(X^2Y^2XY)\mathbb{C}[\textbf{hsop}]$$

as $\mathbb{C}[\textbf{hsop}]$-modules.

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as $\mathbb{C}[\textbf{hsop}]$-modules. Hence,

$$\mathbb{C}[\mathcal{M}_3 \times \mathcal{M}_3]^{\mathrm{GL}_3} \cong \frac{\mathbb{C}[\mathrm{Tr}\, X, \mathrm{Tr}\, Y, \mathrm{Tr}\, X^2, \ldots, \mathrm{Tr}(X^2Y^2), \mathrm{Tr}(XYX^2Y^2)]}{\left(\mathrm{Tr}(XYX^2Y^2)^2 + \cdots\right)}$$

$n = 4$ (Drensky–Sadikova; Drensky–La Scala, 2009)

$$\mathbb{C}[\mathcal{M}_4 \times \mathcal{M}_4]^{\mathrm{GL}_4} \;\cong\; \frac{\mathbb{C}[32 \text{ traces as generators}]}{(\text{relations of degrees } 12,\ 13,\ 14,\ \textbf{?})}.$$

Generators known; **full set of defining relations remained open**.

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The ideal of defining relations of $\mathbb{C}[\mathcal{M}_4 \times \mathcal{M}_4]^{\mathrm{GL}_4}$ satisfies:

- Maximum degree of relations: $\mathbf{20}$.
- Total number of relations: $\mathbf{105}$.
- Only 8 relations generate the whole ideal using the *non-associative* structure.

Let $L = \mathbb{C}\langle x, y\rangle$. Define a bilinear mapping $\{-,-\} : L \times L \to L$ by

$$\{u_1 \cdots u_p,\ v_1 \cdots v_q\} = \sum_{i=1}^{p}\sum_{j=1}^{q} \omega(u_i, v_j) v_1 \cdots v_{j-1} u_{i+1} \cdots u_p u_1 \cdots u_{i-1} v_{j+1} \cdots v_q,$$

where elements $u_1, \dots, u_p, v_1, \dots, v_q$ are either x or y and

$$\omega(x, y) = -\omega(y, x) = 1, \quad \omega(x, x) = \omega(y, y) = 0.$$

Leibniz algebra structure and a Necklace Lie algebra

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Example

$\{xyyy, xxy\} = x^2y^3 - y^2xxy - xy^2xy - yxyxy - xyxyy - xyyxy - xxyyy$

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M. Van den Bergh (2008)

L with this bracket is a **Leibniz** algebra.

Liezation of the Leibniz algebra $L = \mathbb{C}\langle x, y\rangle$

- $\mathrm{Leib}(L) = \mathrm{Span}\{\{a, a\} \mid a \in L\}$ is called the Leibniz kernel.
- In fact, $\mathrm{Leib}(L) = \mathrm{Span}\{\{a, b\} + \{b, a\} \mid$ monomials $a, b \in L\}$.

It is the **universal** Lie quotient of L:

the liezation functor $L \mapsto L/\mathrm{Leib}(L)$ is left adjoint to the inclusion $\mathrm{Lie} \hookrightarrow \mathrm{Leib}$, so that every Leibniz algebra morphism from L to a Lie algebra factors uniquely through the projection $L \to L/\mathrm{Leib}(L)$.

Commutator quotient space

$\mathrm{Com}(L) = \{ab - ba \mid a, b \in L\}$ - called commutator space and $\mathrm{Com}(L) \supseteq \mathrm{Leib}(L)$.

The quotient algebra

$$\mathcal{N} = L/\mathsf{Com}(L)$$

with the induced bracket is a **Lie** algebra, called **Necklace Lie algebra**.

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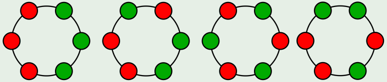
Classes: $x^3y^3, \quad x^2y^2xy, \quad (xy)^3, \quad x^2yxy^2$



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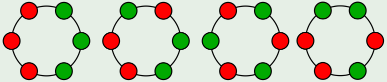
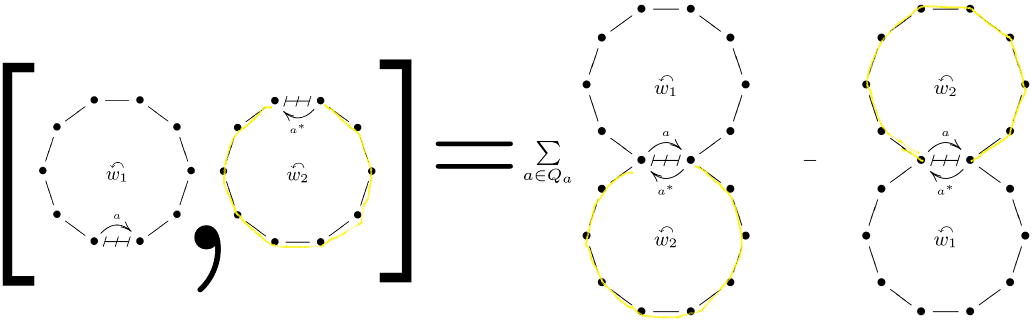
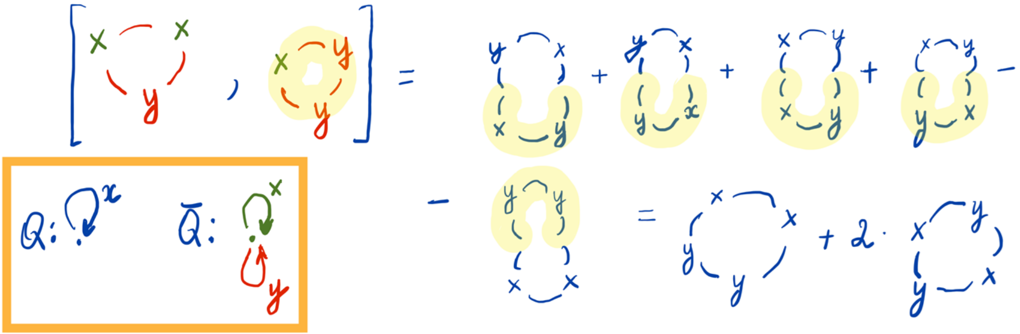


Necklace Lie Algebra $=$ Liezation of Leibniz algebra L?

Lie bracket $[w_1, w_2]$

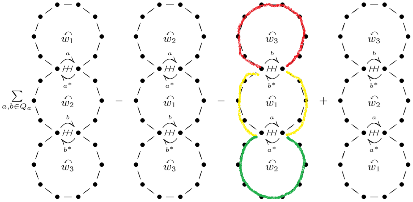


Example of computation



Jacobi identity

$$[[w_1, w_2], w_3]$$



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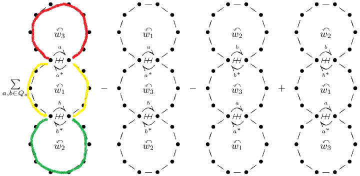
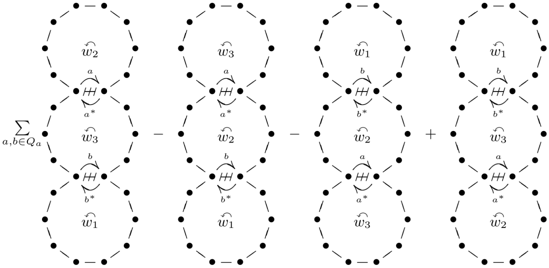


$$[[w_2, w_3], w_1]$$



Basic Properties of the Leibniz algebra $L = \mathbb{C}\langle x, y\rangle$

$$\{a,\, b \cdot c\} = \{a,\, b\} \cdot c + b \cdot \{a,\, c\} \tag{1}$$

$$\{a \cdot b,\, c\} = \{b \cdot a,\, c\} \tag{2}$$

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$$\{a,\, b\cdot c\} = \{a,\, b\}\cdot c + b\cdot\{a,\, c\} \tag{1}$$

$$\{a\cdot b,\, c\} = \{b\cdot a,\, c\} \tag{2}$$

$L = \mathbb{C}\langle x, y\rangle = \bigoplus\limits_{n,m\geq 0} L_{(n,m)}$ is a bi-graded algebra,

where $L_{(n,m)}$ is the subspace spanned by all words with n x's and m y's.

$$\{L_{(a,b)},\, L_{(c,d)}\} \subseteq L_{(a+c-1,\, b+c-1)}$$

Another grading can be considered: $L = \bigoplus_{t \in \mathbb{Z}} V_t$, where

$$V_t = \bigoplus_{n-m=t} L_{(n,m)}.$$

Then the Leibniz bracket respects the grading: $\{V_t, V_s\} \subseteq V_{t+s}$.

Theorem: The necklace Lie $\mathcal{N}$ algebra is the liezation of the Leibniz algebra $L = \mathbb{C}\langle x, y\rangle$.

Sketch of proof: We need to show that $\mathrm{Com}(L) = \mathrm{Leib}(L)$.

Let w be a word of bidegree (n, m), i.e. $w \in L_{(n,m)}$. Then

$$\{xy,\, w\} = (m - n)w.$$

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Part I – Unbalanced case

Let $w \in V_t$ with $t \neq 0$ and let w_1, w_2 be any non-empty subwords of w such that $w = w_1 w_2$. Then

$$\{w_1w_2 - w_2w_1,\, xy\} + \{xy,\, w_1w_2 - w_2w_1\} = -t(w_1w_2 - w_2w_1)$$

which implies that $w_1w_2 - w_2w_1 \in \mathrm{Leib}(L)$.

PART II – Balanced case

As an example, consider a balanced commutator of $x^{n-m}y^n$ and x^m:

$$\begin{aligned} x^n y^n - x^{n-m}y^n x^m &= \frac{1}{n+1}\left(\{x,\, x^n y^{n+1}\} - \{x,\, x^{n-m}y^{n+1}x^m\}\right) \\ &= \frac{1}{n+1}\{x,\, x^n y^{n+1} - x^{n-m}y^{n+1}x^m\} \in \{x, \mathrm{Leib}(L)\} \subseteq \mathrm{Leib}(L). \end{aligned}$$

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One lifts a **balanced** commutator to nonzero weight by adjoining a letter y to create an unbalanced commutator, that already belongs to $\mathrm{Leib}(R)$, and pushes it back down with $\{x, -\}$.

Define a map as follows

$$\mathfrak{tr} : \mathcal{N} \to \mathbb{C}[M_n \times M_n]^{\mathrm{GL}_n}$$

$$\text{Class } x^{k_1}y^{l_1}\cdots x^{k_m}y^{l_m} \mapsto \big((X,Y) \mapsto \mathrm{Tr}(X^{k_1}Y^{l_1}\cdots X^{k_m}Y^{l_m})\big)$$

The map $\mathfrak{tr}$ is a Lie algebra map.[a]

[a] V. Ginzburg, *Non-commutative symplectic geometry, quiver varieties, and operads*, Math. Res. Lett. 2001.

Example of generating a linear equation

Let A and B are traceless versions of X and Y:

$$A = X - \frac{1}{4}\operatorname{Tr}(X)I_4, \quad B = Y - \frac{1}{4}\operatorname{Tr}(Y)I_4.$$

Direct computation shows

$$\{\operatorname{Tr}(B^2), \operatorname{Tr}(A^4B)\} = -4\operatorname{Tr}(A^3B^2) - 4\operatorname{Tr}(A^2BAB).$$

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Whenever a new relation emerges, it is **incorporated** to the candidate ideal of relations, **together** with its Poisson bracket with $\operatorname{Tr}(B^2)$ and $\operatorname{Tr}(A^3)$.

Interpretation by physicists

Robert de Mello Kocha, Antal Jevicki, *Hilbert Space of Finite n Multi-matrix Models*, Journal of High Energy Physics, 2025

- Study of the structure of the space of **gauge-invariant operators** in a d-matrix model at finite n.
- In the dual gauge theory, **trace identities** eliminate redundant states.
- This mechanism highlights a universal feature of quantum gravity – that the true number of independent observables is far smaller than naively expected from *semiclassical* considerations.
- The growth in the number of **secondary invariants** at large n mirrors the expected *scaling of black hole entropy*.
- Primary invariants act freely, naturally giving rise to a Fock space structure.
- A compelling interpretation: secondary invariants may correspond to black hole microstates.

Bukhara, Uzbekistan, 2015

Thank you Manolo for everything and please visit Uzbekistan again!