Intrinsic tensor products and a Ganea-type extension of the five-term exact sequence

Tim Van der Linden[1]

Fonds de la Recherche Scientifique–FNRS
Université catholique de Louvain
Vrije Universiteit Brussel

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[1] Joint work with Bo Shan Deval and Manfred Hartl

1. Ganea's Theorem

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Theorem

Any central extension of groups

$$0 \longrightarrow K \rightarrowtail^{k} B \xrightarrow{f} A \longrightarrow 0 \qquad\qquad [K, B] = 0$$

induces a six-term exact sequence

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- [B. S. Deval, M. Hartl and TVdL, *Intrinsic tensor products and a Ganea-type extension of the five-term exact sequence* (2026), preprint **arXiv:2512.03951v3**]

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Our approach is to **characterise tensor products via commutators**.
It is based on ideas of Manfred Hartl (partly going back to work of, notably, Baues, Eilenberg and Mac Lane) in the context of *functor calculus* and builds on theory established in
[M. Hartl and TVdL, *The ternary commutator obstruction for internal crossed modules*, Adv. Math. **232** (2013)].

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$$A \otimes_{\Bbbk} B \coloneqq \frac{\langle a \otimes b \coloneqq (a, b) \mid a \in A,\, b \in B \rangle_{\Bbbk}}{S}$$

where S is the subspace generated by elements of the form

$$\begin{array}{ccc} (a + a') \otimes b - a \otimes b - a' \otimes b & \text{and} & a \otimes (b + b') - a \otimes b - a \otimes b' \\ \lambda(a \otimes b) - (\lambda a) \otimes b & \text{and} & \lambda(a \otimes b) - a \otimes (\lambda b). \end{array}$$

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This construction is valid for modules over a commutative ring with unit.

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- A (commutative) (associative) $\mathbb{k}$-algebra is **abelian** when $xy = 0$.
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How to make sense of this, formally?

- If $A \otimes_{\Bbbk} B$ is a commutator, then… a commutator *where*?
- How to take the *bilinearity* of the tensor product into account?

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$$\begin{tikzcd} \cdot \arrow[r, tail, "\forall i"] \arrow[d, dotted, two heads, "\exists e"'] & \cdot \arrow[d, two heads, "\forall p"] \\ \cdot \arrow[r, dotted, tail, "\exists m"'] & \cdot \end{tikzcd}$$

- whenever $K \xrightarrow{k} X \underset{s}{\overset{f}{\rightleftarrows}} Y$ where $k = \ker(f)$ and $f \circ s = 1_Y$,

  k and s are jointly extremal-epic. Hence $f = \operatorname{coker}(k)$.

6. The context: semi-abelian categories

[G. Janelidze, L. Márki and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra **168** (2002)]

We need a **zero object**: an object 0, both initial and terminal. Assume:

- finite limits and finite colimits exist;
- normal epimorphisms (= cokernels) are pullback-stable;
- any $p \circ i$ where i is normal mono (= kernel) and p is normal epi can be written as $m \circ e$ where e is normal epi and m is normal mono;

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Some non-abelian examples are:

- all pointed varieties of universal algebras with a group operation: groups, rings, algebras over a reduced operad (Lie, associative, etc.), skew braces;
- loops, Heyting semilattices, cocommutative Hopf algebras, $\boldsymbol{Set}_*^{\mathrm{op}}$.

7. Co-smash products in semi-abelian categories

[A. Carboni and G. Janelidze, *Smash product of pointed objects in lextensive categories*, J. Pure Appl. Algebra **183** (2003)]

$$X + Y \xrightarrow{\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}} X \times Y$$

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Any two objects X and Y in a semi-abelian category $\mathscr{X}$ induce a short exact sequence

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In accordance with the somewhat esoterical *microcosm principle*, the co-smash product $X \diamond Y$ itself behaves as a commutator term.

8. The binary Higgins commutator

Consider K, $L \leqslant X$ in a semi-abelian category $\mathscr{X}$.

$$\begin{tikzcd}
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- Examples: abelian groups in $\boldsymbol{Gp}$; $\Bbbk$-vector spaces in $\boldsymbol{Lie}_{\Bbbk}$ or $\boldsymbol{Alg}_{\Bbbk}$; …
- $\boldsymbol{Ab}(\mathscr{X})$ is an abelian category. In it, all binary co-smashes vanish.

9. Cross-effects and the concept of a linear functor

A functor $F\colon \mathscr{X} \to \mathscr{Y}$ between semi-abelian categories such that $F(0) \xrightarrow{\cong} 0$ is **linear** when $F(X+Y) \xrightarrow{\cong} F(X) \times F(Y)$.

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Any two objects X, Y of $\mathscr{X}$ give rise to a short exact sequence

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The bifunctor $\mathrm{cr}_2(F)\colon \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ is called the **second cross-effect** of F.

- $\mathrm{cr}_2(F)$ vanishes iff F is linear.
- If $F = 1_{\mathscr{X}}$, then $\mathrm{cr}_2(F)(X,Y) = X \diamond Y$.

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Higher cross-effects give rise to a theory of *polynomial functors*.

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A functor $F\colon \mathscr{X} \to \mathscr{Y}$ between semi-abelian categories such that $F(0) \xrightarrow{\cong} 0$ is **linear** when $F(X+Y) \xrightarrow{\cong} F(X) \times F(Y)$.

Examples:

- $1_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}$ is linear iff $\mathscr{X}$ is **linear** (= enriched over ***CMon***) iff $\mathscr{X}$ is abelian.
- When for each Z in $\mathscr{X}$, the functor $(-) \diamond Z\colon \mathscr{X} \to \mathscr{X}$ is linear, $\mathscr{X}$ is said to be a **two-nilpotent** category. (More later.)

***Cross-effects* measure polynomiality of a functor just like *homology* measures exactness.**

Any two objects X, Y of $\mathscr{X}$ give rise to a short exact sequence

$$0 \longrightarrow \mathrm{cr}_2(F)(X,Y) \rightarrowtail F(X+Y) \twoheadrightarrow F(X) \times F(Y) \longrightarrow 0.$$

The bifunctor $\mathrm{cr}_2(F)\colon \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ is called the **second cross-effect** of F.

- $\mathrm{cr}_2(F)$ vanishes iff F is linear.
- If $F = 1_{\mathscr{X}}$, then $\mathrm{cr}_2(F)(X,Y) = X \diamond Y$.

Higher cross-effects give rise to a theory of *polynomial functors*.

(Higher cross-effects $\quad\leftrightsquigarrow\quad$ higher polynomiality degrees.)

10. The ternary co-smash product

The second cross-effect of $(-) \diamond Z \colon \mathscr{X} \to \mathscr{X}$, defined via

$$0 \longrightarrow X \diamond Y \diamond Z \rightarrowtriangle (X + Y) \diamond Z \longrightarrow\!\!\!\triangleright (X \diamond Z) \times (Y \diamond Z) \longrightarrow 0$$

coincides with the **third cross-effect** of $1_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}$, the kernel

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We obtain the **ternary commutator** of $K, L, M \leqslant X$ via

$$\begin{tikzcd} 0 \arrow[r] & K \diamond L \diamond M \arrow[r, tail] \arrow[d, two heads] & K+L+M \arrow[d, "\langle k,\ell,m \rangle"] \\ & {[K, L, M]} \arrow[r, tail] & X \end{tikzcd}$$

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- An object X such that $[X,X,X] = 0$ is called **two-nilpotent**.
  These determine the **two-nilpotent core** $\boldsymbol{Nil}_2(\mathscr{X})$ of $\mathscr{X}$
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- $\boldsymbol{Nil}_2(\mathscr{X})$ is characterised by the condition that all $X \diamond Y \diamond Z$ vanish.

11. So, can $\otimes_{\Bbbk}$ be codified as a co-smash product?

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Hence, $\mathscr{X}$ must be two-nilpotent.

12. The example of groups

The case of groups has been known for a while:

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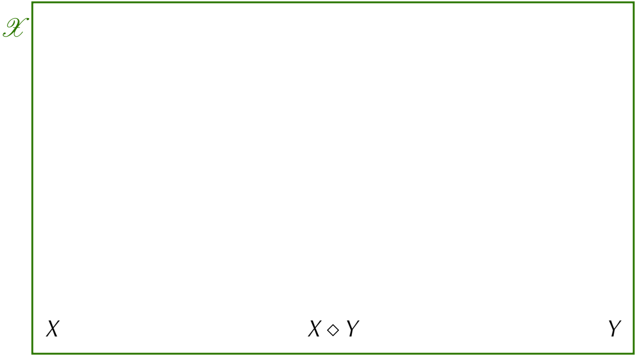
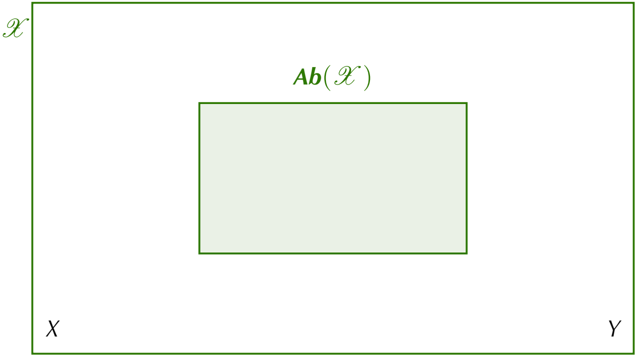
We define the concept of a **bilinear product** in general, look at examples and applications, and then explain that the answer is, essentially, **Yes**.

13. Definition of the **bilinear product**



In $\mathscr{X}$ itself, the co-smash product $X \diamond Y$ is a formal commutator.

13. Definition of the **bilinear product**



In $\boldsymbol{Ab}(\mathscr{X})$, the co-smash product $\mathrm{ab}(X) \diamond_1 \mathrm{ab}(Y)$ is zero.

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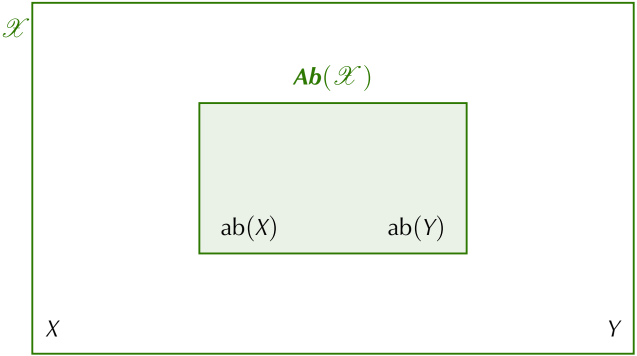

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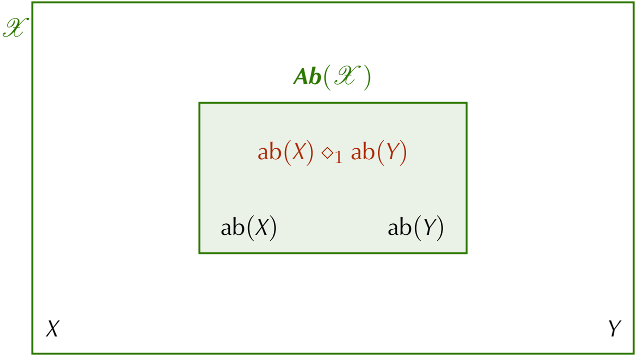

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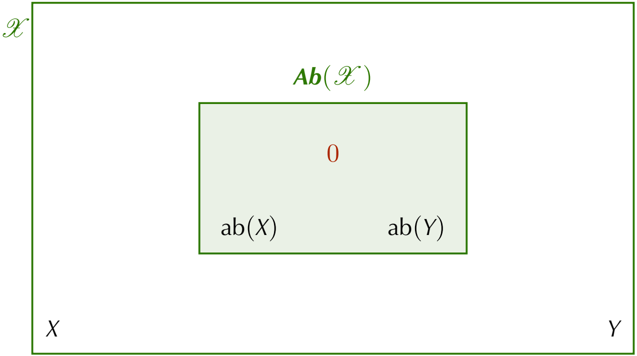

In $\boldsymbol{Ab}(\mathscr{X})$, the co-smash product ab$(X) \diamond_1$ ab(Y) is zero.

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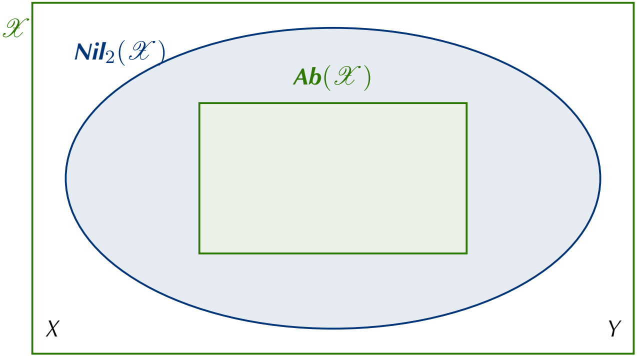

In $\boldsymbol{Nil}_2(\mathscr{X})$, the co-smash product $\mathrm{nil}_2(X) \diamond_2 \mathrm{nil}_2(Y)$ is abelian.

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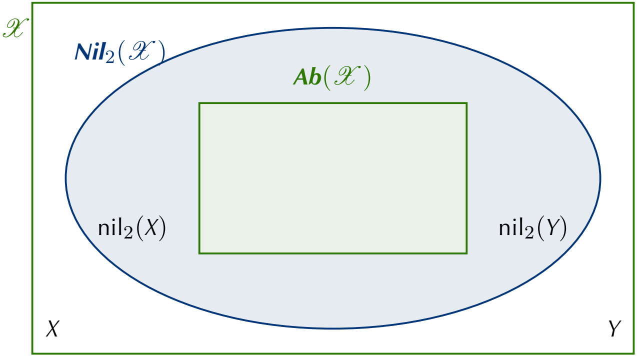

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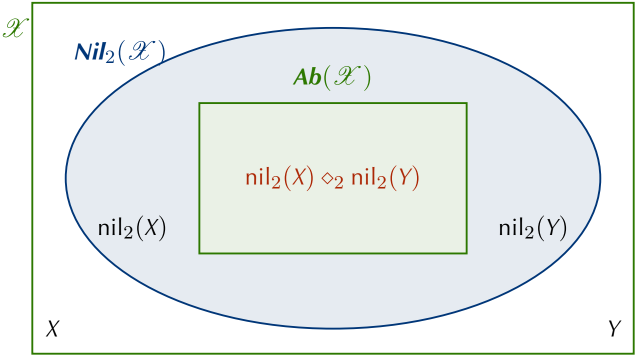

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14. Abelianness of $X \diamond Y$ in $\boldsymbol{Nil}_2(\mathscr{X})$

$$\begin{tikzcd}
(X \diamond Y) \diamond (X \diamond Y) \arrow[r, tail] \arrow[d, twoheadrightarrow] & (X \diamond Y) + (X \diamond Y) \arrow[d, twoheadrightarrow, "\langle 1_{X \diamond Y}, 1_{X \diamond Y} \rangle"] \\
{[X \diamond Y, X \diamond Y]} \arrow[r, tail] & X \diamond Y
\end{tikzcd}$$

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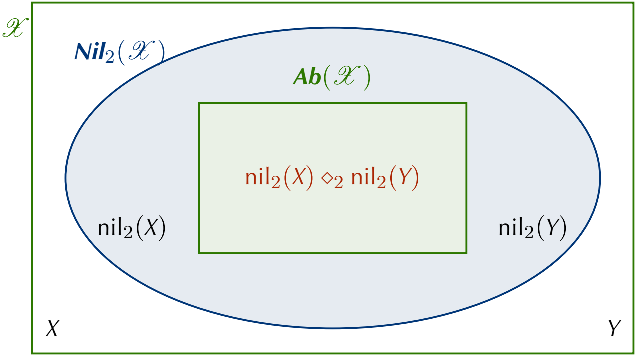
$$

$$
\begin{aligned}
[X \diamond Y, X \diamond Y] = \big[[\iota_1(X), \iota_2(Y)], [\iota_1(X), \iota_2(Y)]\big] &\leqslant [\iota_1(X), \iota_2(Y), \iota_1(X), \iota_2(Y)] \\
&\leqslant [\iota_1(X), \iota_1(X), \iota_2(Y)] = 0
\end{aligned}
$$

(as subobjects of $X + Y$)

15. Definition of the **bilinear product**



In $\boldsymbol{Nil}_2(\mathscr{X})$, the co-smash product $\mathrm{nil}_2(X) \diamond_2 \mathrm{nil}_2(Y)$ is abelian.

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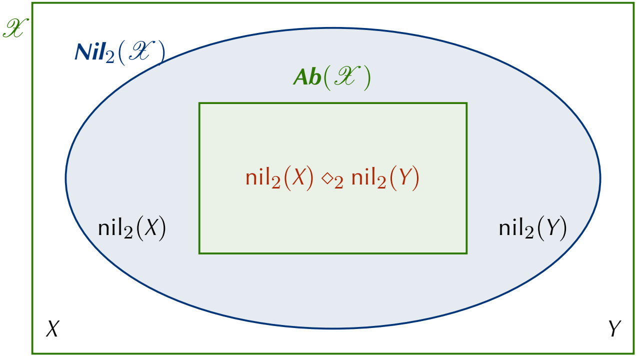

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Definition

Given X and Y in a semi-abelian category $\mathscr{X}$, their **bilinear product** $X \otimes_{\mathscr{X}} Y \in |\boldsymbol{Ab}(\mathscr{X})|$ is the co-smash product $\mathrm{nil}_2(X) \diamond_2 \mathrm{nil}_2(Y)$ in the two-nilpotent core $\boldsymbol{Nil}_2(\mathscr{X})$ of $\mathscr{X}$.

We obtain a functor $\otimes_{\mathscr{X}} : \mathscr{X} \times \mathscr{X} \to \boldsymbol{Ab}(\mathscr{X}) : (X, Y) \mapsto X \otimes_{\mathscr{X}} Y$.

16. Examples

As an operation on an abelian category $\mathscr{A} = \boldsymbol{Ab}(\mathscr{X})$, the bilinear product depends on $\mathscr{X}$.

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As an operation on an abelian category $\mathscr{A} = \boldsymbol{Ab}(\mathscr{X})$, the bilinear product depends on $\mathscr{X}$. In the table, A, B are abelian objects, R is a commutative unitary ring, and G is a group.

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$\boldsymbol{Pt}_G(\boldsymbol{Gp})$	G-actions	$\boldsymbol{Mod}_{\mathbb{Z}[G]}$	$A \otimes_{\mathbb{Z}} B$ with appropriate G-action
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Suppose J is a unit for $\otimes_{\mathscr{X}}$ on $\boldsymbol{Ab}(\mathscr{X}) = \boldsymbol{Vect}_{\Bbbk}$ when $\mathscr{X} = \boldsymbol{Alg}_{\Bbbk}$; then in $\boldsymbol{Vect}_{\Bbbk}$ we have

$$\Bbbk \cong \Bbbk \otimes_{\mathscr{X}} J = (\Bbbk \otimes_{\Bbbk} J) \oplus (J \otimes_{\Bbbk} \Bbbk) \cong J \oplus J$$

so that $\dim(J) = \frac{1}{2}$. Impossible! Hence units cannot exist in general.

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[T. Ganea, *Homologie et extensions centrales de groupes*, C. R. Acad. Sci. Paris Sér. A-B **266** (1968)]
[T. Everaert and TVdL, *Baer invariants in semi-abelian categories II: Homology*, Theory Appl. Categ. **12** (2004)]

Theorem

Any central extension

$$0 \longrightarrow K \overset{k}{\rightarrowtriangle} B \overset{f}{\longrightarrow\!\!\triangleright} A \longrightarrow 0 \qquad\qquad [K,B]=0$$

in a semi-abelian algebraically coherent category with enough projectives induces a six-term exact sequence

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- This captures the known results for groups, (pre)crossed modules, associative algebras, Lie and Leibniz algebras due to Arias, Casas, Ganea, Ladra and Pirashvili.

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Note that the category $\mathscr{X}$ depends on the symmetry isomorphisms $\tau_{A,B} \colon A \otimes B \to B \otimes A$.

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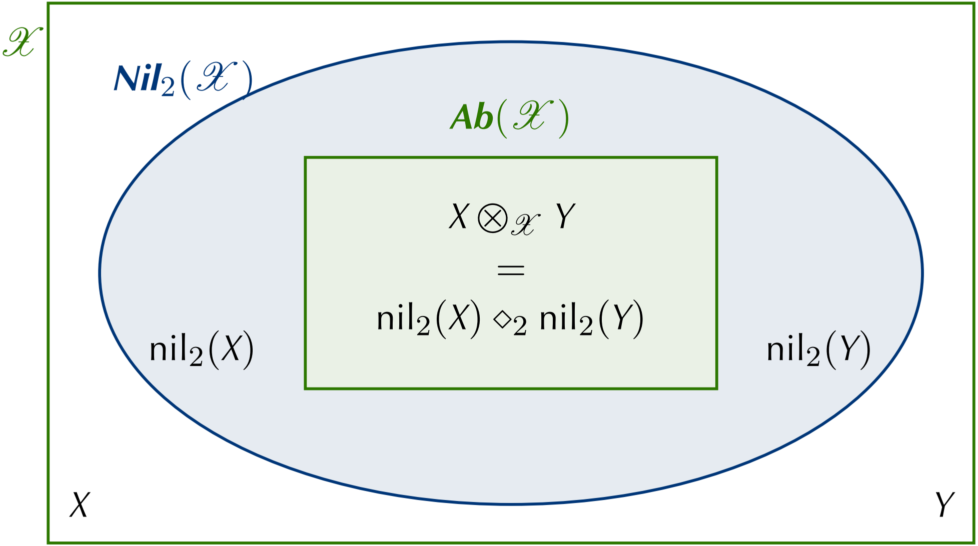
A substantial research project awaits us, where we work out the functor calculus related to all this, extending for instance [S. Eilenberg and S. Mac Lane, *On the groups $H(\Pi, n)$ II: Methods of computation*, Ann. of Math. (2) **60** (1954)]
The concepts of a *quadratic* functor and higher-order *polynomial* functors lead to prospective results including (in the very long run) a semi-abelian version of the *Baker–Campbell–Hausdorff formula*.

Happy birthday, Manolo!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