Computing the regularity of the deficiency modules

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Categorical, homological, and computational methods in Algebra

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Dedicated to Manuel Ladra
for his 70th birthday

An excellent mathematician

An exemplary colleague

And a better friend

Based on joint work with Alberto F. Boix

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Castelnuovo-Mumford regularity

Schenzel's graded deficiency modules

Main result

Some consequences

CASTELNUOVO–MUMFORD REGULARITY

Some notation to start

- $\mathbb{K}$ field, $A = \mathbb{K}[X_1, \ldots, X_r]$, $\deg(X_i) = 1$.
- $I \subset A$ homogeneous ideal.
- $I = (f_1, \ldots, f_n)$, f_j homogeneous polynomials.
- M a finitely generated graded A–module.

An invariant attached to homogeneous ideals

Claim

If $f_1, \ldots, f_n$ is a minimal system of generators for I, then

$$d_I := \max_{1 \leq j \leq n} \deg(f_j)$$

is an invariant of I.

Problem

d_I is hard to handle.

Another perspective

Study the degrees of elements required to generate all the syzygies of I.

Hilbert Syzygy Theorem (graded)

Theorem

Any f.g. graded A–module M has a finite graded free resolution

$$\dots \longrightarrow F_i \xrightarrow{\phi_i} F_{i-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

of length at most r, where $F_i = \bigoplus A(-j)^{\beta_{ij}}$.

We can take a minimal free resolution

In fact, any f.g. graded A–module M admits a unique (up to isomorphism) minimal finite graded free resolution

$$\ldots \longrightarrow F_i \xrightarrow{\phi_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

where minimal means that each ϕ_i sends a basis of F_i to a minimal set of generators of the image of ϕ_i.

Equivalently, after fixing basis, each entry of the matrix representing ϕ_i belongs to $(X_1, \ldots, X_r)$.

The Betti numbers

For each pair (i, j) the number β_{ij} is the minimal number of generators of degree j of the i–th syzygy module of M.

(For instance, β_{0j} is the number of elements of degree j in a minimal system of homogeneous generators of M.)

It is a finite collection of finite numbers uniquely determined for any minimal free resolution of M.

They are the Betti numbers of M.

We may think that the complexity of the minimal free resolution is as higher as higher are the degrees of the generators of the syzygies of M.

In general, they increase at least in one at each step of the resolution.

For instance, we say that M admits a d-linear resolution if $\beta_{ij} = 0$ for pair such that $j \neq d + i$. In particular, M is generated by elements of degree d,

A typical example is the Koszul complex of the ideal $(X_1, \ldots, X_n)$, which is a 1-linear resolution of $(X_1, \ldots, X_n)$.

Defining the regularity: step 1

Definition

Set b_i as the maximum of the degrees of the generators of F_i.

Important remark

We work with the sequence $\{b_i - i : \ i \geq 0\}$.

- We say that M is m–regular for some $m \in \mathbb{Z}$ if

$$b_i - i \leq m \text{ for all } i.$$

If M is m–regular, then M is generated by elements of degrees $\leq m$.

The regularity

Definition (Castelnuovo–Mumford)

Set

$$\mathrm{reg}(M) := \min\{m \in \mathbb{Z} : \ M \text{ is } m\text{–regular}\}.$$

In terms of Betti numbers we have that

$$\mathrm{reg}(M) = \max\{j \in \mathbb{Z} \mid \beta_{i(i+j)} \neq 0 \text{ for some } i\}.$$

Regularity: some remarks

- If M is free, then $\operatorname{reg}(M)$ agree with the maximum of the degrees of a minimal set of generators of M.
- If M has finite length, then

$$\operatorname{reg}(M) = \max\{n : \ M_n \neq 0\}.$$

- If M has a d-linear resolution then $\operatorname{reg}(M) = d$.

A good property of the regularity

- Let
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$
be a short exact sequence of f.g. graded A–modules.
- Then, we have that
$$\mathrm{reg}(M) \leq \max\{\mathrm{reg}(M'), \mathrm{reg}(M'')\}.$$
- Equality holds, for instance, when either M' is of finite length or $M = M' \oplus M''$.

As a consequence we get:

- M graded f.g. A–module.
- Assume M admits a finite filtration by f.g. graded A–modules:

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \ldots \subset F_\ell = M.$$

- Then, we have

$$\operatorname{reg}(M) \leq \max_{0 \leq j \leq \ell} \operatorname{reg}\left(\frac{F_j}{F_{j-1}}\right).$$

SCHENZEL'S GRADED DEFICIENCY MODULES

Deficiency modules: definition

- Let M a finitely generated graded A–module.

Definition (P. Schenzel, 1982)

For each $j \in \mathbb{Z}$, one defines the graded j-th deficiency module of M as

$$K^j(M) := \mathrm{Ext}_A^{r-j}(M, A(-r)).$$

At first instance, these modules measure the failure of the Cohen-Macaulay property of M, since M is Cohen-Macaulay if and only if $K^j(M) = 0$ for any $j \neq \dim(M)$.

Deficiency modules: some remarks

- $K^j(M) = 0$ if either $j < 0$ or if $j > \dim(M)$.
- For any $\mathfrak{p} \in \mathrm{Supp}(M)$ and any j,

  $$K^j(M)_{\mathfrak{p}} \cong K^{j-\dim(A/\mathfrak{p})}(M_{\mathfrak{p}}).$$

  In particular, for any j,

  $$\dim(K^j(M)) \leq j.$$

- For $j = \dim(M)$, $K^j(M) = K(M)$ is the graded canonical module of M.

Hilbert series and deficiency modules

We are interested in the study of the regularity of the deficiency modules. Why? This is an example:

- Assume that $R := A/I = \dfrac{\mathbb{K}[X_1, \dots, X_r]}{I}$, $I \subset A$ homogeneous ideal.
- R is a standard graded algebra. In particular,

$$R = \bigoplus_{i \geq 0} R_i,\ \dim_{\mathbb{K}}(R_i) < \infty.$$

- Its Hilbert series is

$$H_R(T) := \sum_{i \geq 0} (\dim_{\mathbb{K}}(R_i)) T^i.$$

- It is known that
$$H_R(T) = \frac{h_0 + h_1 T + \ldots + h_s T^s}{(1-T)^d}.$$
- $(h_0, \ldots, h_s)$ is called the h–vector of R.

Classical question

Under which conditions $h_i \geq 0$ for any $i \geq 0$?

If R is Cohen-Macaulay this is true (in fact, in this case $h_i > 0$ for any $i \geq 0$).

h–vector and regularity of deficiency modules

Theorem (Murai and Terai (2009))

If

$$\mathrm{reg}(K^j(R)) \leq j - r \text{ for all } 0 \leq j \leq d-1,$$

then

$$h_i \geq 0 \text{ for all } i \leq r.$$

Remark

In 2024, Ma, Dao, and Varbaro extended this statement replacing R by any f.g. graded R–module M.

Some results

There are some general bounds...

- $A = \mathbb{K}[X_1, \ldots, X_r]$ with $r \geq 2$.
- $I \subset A$ homogeneous containing no linear form.

Theorem (Hoa and Hyry (2006))

We have that

$$\operatorname{reg}(K^j(A/I)) < \begin{cases} 4(\operatorname{reg}(I))^{r-1} - 4(\operatorname{reg}(I))^{r-2}, & \text{if } j = 1, \\ (2\operatorname{reg}(I))^{r\cdots(r+j-1)2^{\frac{j(j-1)}{2}}}, & \text{if } j \geq 2. \end{cases}$$

The next one will play an important role in main result:

- $I \subset A$ graded.
- Assume A/I is Cohen–Macaulay.

Theorem (Ha and Hoa (2008))

We have that

$$\mathrm{reg}(K(A/I)) = \dim(A/I).$$

- $I \subset A$ monomial.

Theorem (Kummini and Murai (2011))

We have

$$\mathrm{reg}(K^j(A/I)) \leq j.$$

Remark

One of our motivations was to extend this bound to other classes of ideals.

MAIN
RESULT

General setting

- $I \subseteq A$ ideal, $I = I_1 \cap \ldots \cap I_t$ a decomposition.

We want to compute the regularity of the deficiency modules $K^j(A/I)$ in terms of this decomposition.

For that, we will consider a finite poset P constructed in the following way:

- P: all different sums of I_i's ordered by reverse inclusion.
- $p \in P$ is just I_p, sum of certain I_i's.
- $\widehat{P} := P \cup \{0_{\widehat{P}}, 1_{\widehat{P}}\}$, with $I_{0_{\widehat{P}}} = A, I_{1_{\widehat{P}}} = I$.

A picture

$$
\begin{tikzcd}
 & I & \\
I_1 \arrow[ur, dotted] & & I_2 \arrow[ul, dotted] \\
 & I_1 + I_2 \arrow[ul] \arrow[ur] & \\
 & A \arrow[u, dotted] &
\end{tikzcd}
$$

Inverse limit rings

Now, taking quotients, we get an inverse system of A-algebras and take the limit:

- $(A/I_p \longrightarrow A/I_q)_{p \geq q}$ inverse system of A-algebras

$$A_P := \lim_{p \in P} A/I_p.$$

In general, we cannot expect that $A/I \simeq A_P$, but this happens in the cases we are interested in.

A picture

$$\begin{tikzcd}
 & A/I \arrow[dl, dotted] \arrow[dr, dotted] & \\
A/I_1 \arrow[dr] & & A/I_2 \arrow[dl] \\
 & A/(I_1 + I_2) \arrow[d, dotted] & \\
 & (0) &
\end{tikzcd}$$

Order complex of a poset and homology

- P finite poset, $\widehat{P} := P \cup \{0_{\widehat{P}}, 1_{\widehat{P}}\}$.
- $Q \subseteq P$ subposet, $\Delta(Q)$ order complex of Q.
- $(p, 1_{\widehat{P}}) := \{q \in P : \ q > p\}$.

$$\widetilde{H}_s((p, 1_{\widehat{P}}); \mathbb{K}) := \widetilde{H}_s(\Delta((p, 1_{\widehat{P}})); \mathbb{K})$$

Main result: some notation

- Fix $j \in \mathbb{Z}$.
- For each $p \in P$, set

$$m_{r-j,p} := \dim_{\mathbb{K}}(\widetilde{H}_{j-\dim(A/I_p)-1}((p, 1_{\widehat{P}}); \mathbb{K})).$$

- Set

$$S_j := \{p \in P : \ j \geq \dim(A/I_p), \ m_{r-j,p} \neq 0\}.$$

Main result: the assumptions we need

- $A = \mathbb{K}[X_1, \ldots, X_r]$, $I \subset A$ homogeneous.
- P poset given by all the different sums of the primary components of I ordered by reverse inclusion.
- First key hypothesis P is a subset of a distributive lattice of ideals (with respect to the sum and intersection).
- Second key hypothesis For any $p \in P$, A/I_p is Cohen–Macaulay.
- Third key hypothesis If $p < q$, then $\mathrm{ht}(I_q) < \mathrm{ht}(I_p)$.

Main result: the statement

Theorem (Boix and Zarzuela (2025))

Under the previous assumptions,

$$\operatorname{reg}(K^j(A_P)) \leq \max_{p \in S_j} \dim(A/I_p) \leq j.$$

Key ingredient of the proof

- Fix $j \in \mathbb{Z}$.
- Suppose that our previous assumptions hold.
- Then, $K^j(A_P)$ admits a finite filtration

$$0 = H_{-1}^{r-j} \subset H_0^{r-j} \subset H_1^{r-j} \subset \ldots \subset H_t^{r-j} = K^j(A_P)$$

such that

$$\frac{H_k^{r-j}}{H_{k-1}^{r-j}} \cong \bigoplus_{\substack{p \in P \\ j-k=\dim(A/I_p)}} K^{\dim(A/I_p)}(A/I_p)^{\oplus m_{r-j,p}},$$

where

$$m_{r-j,p} := \dim_{\mathbb{K}}(\widetilde{H}_{j-\dim(A/I_p)-1}((p, 1_{\widehat{P}}); \mathbb{K})).$$

Once we have this filtration we just need to use:

- The good behaviour of the regularity for finite filtrations.
- The value obtained by Ha–Hoa for the Cohen-Macaulay case.

How you obtain this filtration?

Under our assumptions, this filtration comes from the degeneration at the second page of a certain spectral sequence.

- This spectral sequence involves the derived functors of the colimit.
- To compute this derived functors we use the so named homological Roos complex.
- And in the computation of the homology of this complex is where the reduced homologies

$$\widetilde{H}_s((p, 1_{\widehat{P}}); \mathbb{K})$$

appear.

Why the assumptions?

- First key hypothesis P is a subset of a distributive lattice of ideals.

This is to guarantee that the involved inverse system is acyclic with respect to the limit and that its limit is A/I.

- Second key hypothesis For any $p \in P$, A/I_p is Cohen–Macaulay.

- Third key hypothesis If $p < q$, then $\mathrm{ht}(I_q) < \mathrm{ht}(I_p)$.

These two hypothesis allow to compute explicitly the second page of the spectral sequence and guarantee the degeneration of the spectral sequence.

SOME CONSEQUENCES

Recovering Kummini–Murai's upper bound

Monomial ideals have a primary decomposition that satisfies the assumptions of our result.

- Hence, when I is a monomial ideal, we recover with a completely new proof Kummini–Murai's upper bound.
- In some examples, our bound is sharper than Kummini–Murai's one even in the monomial case.

Example comparing bounds

- $I = (XZ, XW, YZ, YW) \subset A = \mathbb{K}[X, Y, Z, W]$.
- $I = (X, Y) \cap (Z, W)$, and the attached poset is equal to the one we have given before.
- A/I is a 2-dimensional ring, with depth 1.
- Hence, $K^0(A/I) = 0$, $K^1(A/I) \neq 0$, and $K^2(A/I) \neq 0$.

- Our bound says

$$\operatorname{reg}(K^1(A/I)) \leq 0.$$

- Kummini–Murai's bound says

$$\operatorname{reg}(K^1(A/I)) \leq 1.$$

- Actually (computed with Macaulay2), we have

$$\operatorname{reg}(K^1(A/I)) = 0.$$

Binomial edge ideals: definition

- G finite simple graph on n vertices.
- $\mathbb{K}$ field, $A = \mathbb{K}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$.

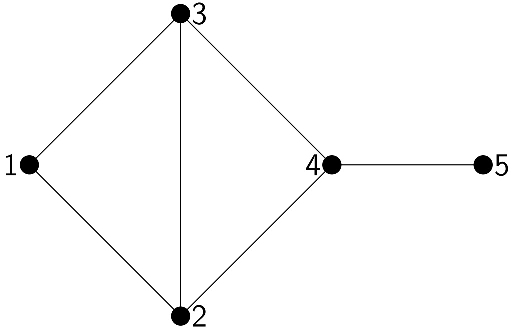
Definition

The binomial edge ideal J_G is the ideal of A generated by

$$X_i Y_j - X_j Y_i,$$

where $i < j$ and $\{i, j\}$ is an edge of G.

Binomial edge ideal: specific example



$$J_G = (X_1Y_2 - X_2Y_1, X_1Y_3 - X_3Y_1, X_2Y_3 - X_3Y_2,$$
$$X_2Y_4 - X_4Y_2, X_3Y_4 - X_4Y_3, X_4Y_5 - X_5Y_4).$$

Binomial edge ideals: Àlvarez Montaner's poset

- G finite simple graph on n vertices.
- J_G binomial edge ideal.
- $J_G = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_t$ minimal prime decomposition.
- P poset given by all the possible different sums of the $\mathfrak{p}_i$'s ordered by reverse inclusion.

Q poset constructed iteratively as follows.

- Q contains all the prime ideals of P.
- For each non–prime ideal I_q, add to Q all the possible prime ideals of the poset P_{I_q}.
- Repeat this process every time you find a non–prime ideal.

Main result for binomial edge rings

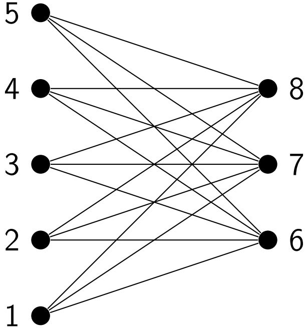
- G finite simple graph on n vertices.
- $\mathbb{K}$ field, $A = \mathbb{K}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$.
- Assume Q is a distributive lattice of ideals.

Then, we have

$$\operatorname{reg}(K^j(A/J_G)) \leq \max_{p \in S_j} \dim(A/I_p) \leq j.$$

One binomial edge ring where our assumptions hold

- G be the complete bipartite graph $K_{5,3}$.



We have that:

- $K^j(A/J_G) \neq 0$ for $j \in \{5, 6, 7, 9, 10\}$.
- It turns out that

$$\begin{cases} 4 = \operatorname{reg}(K^5(A/J_G)) \leq 4 < 5, \\ 6 = \operatorname{reg}(K^7(A/J_G)) \leq 6 < 7. \end{cases}$$

Toric face rings: definition

- $\Sigma \subseteq \mathbb{R}^n$ rational pointed fan.
- $\mathcal{M} := \{M_C \subseteq C \cap \mathbb{Z}^n : \; C \in \Sigma\}$ monoidal complex.

$$\mathbb{K}[\mathcal{M}] := \left\{ \sum_{\mathrm{v} \in |\mathcal{M}|} a_{\mathrm{v}} \mathrm{x}^{\mathrm{v}} : \; a_{\mathrm{v}} \neq 0 \text{ for finite } \mathrm{v}'s \right\},$$

where

$$\mathrm{x}^{\mathrm{v}} \cdot \mathrm{x}^{\mathrm{v}'} := \begin{cases} \mathrm{x}^{\mathrm{v}+\mathrm{v}'}, \; \text{if } \mathrm{v}, \mathrm{v}' \in M_C \text{ for some } C, \\ 0, \; \text{otherwise.} \end{cases}$$

Toric face rings: general examples

- (Stanley, 1987) If $M_C = C \cap \mathbb{Z}^n \; \forall C$, $\mathbb{K}[\mathcal{M}] = \mathbb{K}[\Sigma]$.
- If $\Sigma = \mathrm{Fan}(C)$, $\mathbb{K}[\mathcal{M}] = \mathbb{K}[M_C]$.
- If C has exactly $\dim(C)$ generators and M_C is generated by $\dim(C)$ elements $\forall C$, then $\mathbb{K}[\Sigma]$ is a Stanley–Reisner ring.

Toric face rings: specific example

$O = (0,0,0)$, $A_1 = (2,0,0)$, $A_2 = (0,2,0)$, $A_3 = (0,0,2)$, $A_4 = (1,1,0)$.

$$\Sigma := \mathrm{Fan}\left(OA_1A_2,\ OA_1A_3,\ OA_2A_3\right).$$

$$\mathcal{M} := \{\langle A_1, A_2, A_4\rangle,\ \langle A_1, A_3\rangle,\ \langle A_2, A_3\rangle\}$$

$$\mathbb{K}[\mathcal{M}] = \frac{\mathbb{K}[X_1, X_2, X_3, X_4]}{(X_1X_2 - X_4^2, X_3X_4)}$$

Upper bound for some toric face rings

- $\Sigma \subseteq \mathbb{R}^n$ rational pointed fan.
- $\mathcal{M} := \{M_C \subseteq C \cap \mathbb{Z}^n : \; C \in \Sigma\}$ Cohen–Macaulay monoidal complex.

Then,

$$\operatorname{reg}(K^j(\mathbb{K}[\mathcal{M}])) \leq \max_{p \in S_j} \dim(A/I_p) \leq j.$$

Remark

To the best of our knowledge, this upper bound is completely new.

Bound for toric face rings: an example

- Set

$$A/I = \frac{\mathbb{Q}[X_1, X_2, X_3, X_4, X_5, X_6]}{(X_1X_2 - X_4^2, X_2X_3 - X_5^2, X_4X_6, X_5X_6)}$$

This is a 3-dimensional ring with depth 2 (hence not Cohen-Macaulay) but a Cohen–Macaulay monoidal complex

- In this case,

$$0 = \operatorname{reg}(K^2(A/I)) \leq 2.$$

- This shows that our bound in general is not an equality.

And this is a good moment to finish.

I would only like to conclude by wishing Manolo that he can keep enjoying life and his family just as he has up to now, and continue being the great friend and colleague of all of us he has always been.

Thank you very much.