# Infinite dimensional superalgebras.

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## The ManoloFest

## 1. Superalgebras.

Def. A superalgebra is an algebra graded by the cyclic group of order 2

$$A = A_{\bar{0}} + A_{\bar{1}}$$

$$A_{\bar{0}} A_{\bar{0}} \subseteq A_{\bar{0}}, \quad A_{\bar{0}} A_{\bar{1}} + A_{\bar{1}} A_{\bar{0}} \subseteq A_{\bar{1}}, \quad A_{\bar{1}} A_{\bar{1}} \subseteq A_{\bar{0}}$$

Examples. 1) $F$ a field, $A = M_{m+n}(F)$ matrices of size $m+n$

$$M_{m+n}(F) = \begin{array}{cc} \left(\begin{array}{c|c} & O \\ \hline O & \end{array}\right) & \begin{array}{c} m \\ n \end{array} \end{array} + \left(\begin{array}{c|c} O & \\ \hline & O \end{array}\right)$$

2) $Q(n) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| \, a, b \text{ are } n \times n \text{ matrices} \right\}$

$$Q(n) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\} + \left\{ \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \right\}$$

C. T. C. Wall (1956). $F$ algebraically closed field, $A$ a finite dimensional simple associative superalgebra:

$$M_{m+n}(F) \text{ or } Q(n).$$

3) Grassmann Algebra.

$$G_n = \langle 1, \xi_1, \dots, \xi_n \mid \xi_i^2 = 0, \xi_i \xi_j + \xi_j \xi_i = 0 \rangle$$

Basis: $1, \xi_{i_1} \cdots \xi_{i_k}, \; i_1 < \dots < i_k$

$$G_n = G_{n\bar{0}} + G_{n\bar{1}}$$

$\xi_{i_1} \cdots \xi_{i_k}$ is even if $k$ is even, odd if $k$ is odd.

$$G = G_\infty = \bigcup_{n \geq 1} G_n, \quad G = G_{\bar{0}} + G_{\bar{1}}$$

Let $A = A_{\bar{0}} + A_{\bar{1}}$ be a superalgebra.

$$A_{\bar{0}} \otimes G_{\bar{0}} + A_{\bar{1}} \otimes G_{\bar{1}} < A \otimes G$$

Grassmann envelope

$\mathcal{V}$ = variety of algebras

i.e. a class of algebras defined by identities.

Examples. Associative algebras

$$(xy)z = x(yz)$$

Commutative algebras

$$xy = yx$$

Lie algebras

$$[x,y] = -[y,x]$$

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$$

Jordan algebras

$$xy = yx$$

$$(x^2 y)x = x^2(yx)$$

Def. A is a $\mathcal{V}$-superalgebra if $G(A) \in \mathcal{V}$

$\Leftrightarrow$ a bunch of graded identities.

Commutativity: $xy = (-1)^{|x|\cdot|y|}\, yx$

Lie superalgebras

$$[x,y] = -(-1)^{|x|\cdot|y|}[y,x]$$

$$[[x,y],z] + (-1)^{|x|(|y|+|z|)}[[y,z],x] + (-1)^{|z|(|x|+|y|)}[[z,x],y] = 0$$

Associative superalgebras =

$\mathbb{Z}/2\mathbb{Z}$ graded associative algebras

Jordan algebras

$$xy = (-1)^{|x|\cdot|y|} yx$$

(exercise !)

Grassmann algebra is a commutative superalgebra, but not a commutative algebra.

A $\mathbb{Z}/2\mathbb{Z}$-graded commutative domain $A = A_{\bar{0}} + A_{\bar{1}}$, $A_{\bar{1}} \neq (0)$, char $F \neq 2$, is a commutative algebra, but not a commutative superalgebra.

## Some Examples.

$A = A_{\bar{0}} + A_{\bar{1}}$ associative superalgebra.

$A^{(-)} = (A, [a,b] = ab - (-1)^{|a|\cdot|b|} ba)$ Lie superalgebra

$A^{(+)} = (A, a \circ b = ab + (-1)^{|a|\cdot|b|} ba)$ Jordan superalgebra.

$* : A \to A$, $A_{\bar{i}}^{*} = A_{\bar{i}}$ is a superinvolution

if $\forall\, a, b \in A_{\bar{0}} \cup A_{\bar{1}}$

$(a^*)^* = a$, $(ab)^* = (-1)^{|a|\cdot|b|} b^* a^*$

Skew-symmetric elements : Lie superalgebra

symmetric elements : Jordan superalgebra

V. G. Kac (1976) : classification of finite dimensional ~~Lie super~~ simple Lie superalgebras / algebraically closed fields of zero characteristic;

V. G. Kac (1979) : classification of finite dimensional simple Jordan superalgebras / algebraically closed fields of zero characteristic.

+ one series added by I. Kantor (1988).

## Infinite Dimensional Superalgebras.

$F[t, t^{-1}]$ Laurent polynomials.

$\mathrm{Der}\, F[t, t^{-1}] = \left\{ f(t) \frac{d}{dt} \right\}$ Witt algebra or the (centerless) Virasoro algebra.

Basis : $e_i = t^{i+1}\frac{d}{dt}$, $i \in \mathbb{Z}$,

$$[e_i, e_j] = (j-i)\, e_{i+j}$$

Central Extensions. $L$ is perfect i.e. $L = [L, L]$. Let $L, \tilde{L}$ be perfect Lie (super)algebras, $\varphi : \tilde{L} \to L$ is an epimorphism, $\ker \varphi \subseteq$ Center of $\tilde{L}$. Then $\varphi$ is a central extension.

I. Schur (1904): for any perfect $L$ there exists a unique universal central extension

$$\hat{L} \to L$$

$\forall$ central extension $L' \to L$ there exists $\hat{L} \to L'$ :

$$\begin{tikzcd} \hat{L} \arrow[r] \arrow[dr] & L' \arrow[d] \\ & L \end{tikzcd}$$ is commutative.

The universal central extensions of $\mathrm{Der}\, F[t, t^{-1}]$ is

$$Vir = \sum_{i \in \mathbb{Z}} F e_i + F ¢, \quad ¢ \text{ is central}$$

$$[e_i, e_j] = (j-i) e_{i+j} + \delta_{i,-j} \frac{j^3 - j}{12} ¢$$

## Brackets.

$A$ associative commutative superalgebra.

$$[\,,\,] : A \times A \to A$$

bilinear even mapping, "even" means that $[A_{\bar{i}}, A_{\bar{j}}] \subseteq A_{\overline{i+j}}$.

$[\,,\,]$ is a Poisson bracket if

1) $(A, [\,,\,])$ is a Lie superalgebra,

2) $[ab, c] = a[b, c] + (-1)^{|b| \cdot |c|} [a, c] b$.

Examples. 1) $A = F[p_1, \dots, p_n, q_1, \dots, q_n]$;

$f, g \in A$

$$[f,g] = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right);$$

2) $A = G(n)$, $[f,g] = \sum_{i=1}^{n} (-1)^{|f|} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i}$

For any Poisson bracket $[a, 1] = 0$ $\forall a$.

An even bilinear bracket $[\,,\,]$ is called a contact bracket if

1) $(A, [\,,\,])$ is a Lie superalgebra,

2) $D(a) = [a, 1]$ is a derivation,

3) $[ab, c] = a[b, c] + (-1)^{|b| \cdot |c|} [a, c] b + ab\, D(c)$.

Examples. 1) $A = F[t]$, $[f(t), g(t)] = f'g - fg'$;

2) $A = F[p_1, \dots, p_n, q_1, \dots, q_n, t]$

$$[f, g] = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - \frac{1}{2} \left( \frac{\partial f}{\partial t} g - f \frac{\partial g}{\partial t} \right)$$

(Poisson bracket) $\otimes$ (Poisson bracket) = Poisson bracket

Let $A$ be a contact superalgebra, $B$ a Poisson algebra.

Can we define a contact bracket on

$A \otimes B$ ?

Theorem (C. Martinez - Z. Zhang - E. Z.) Let $B = \bigoplus_{i \geq 0} B_i$ be a graded Poisson superalgebra,

$[B_i, B_j] \subseteq B_{i+j-k}$, $\forall i, j$, $k \neq 0$.

Then

$$[a_1 \otimes b_1, a_2 \otimes b_2] = (-1)^{|b_1|\cdot|a_2|} [a_1, a_2] \otimes b_1 b_2$$
$$+ (-1)^{|b_1|\cdot|a_2|} a_1 a_2 \otimes [b_1, b_2] +$$
$$(-1)^{|b_1|\cdot|a_2|} \cdot \frac{1}{K} \left( \deg(b_1)\, a_1 D(a_2) - \deg(b_2)\, D(a_1) a_2 \right) \otimes b_1 b_2$$

is a contact bracket on $A \otimes B$.

## Superconformal Algebras.

Vir is useful. Does it have useful superextensions?

Neveu - Schwarz (1971), P. Ramond (1971)

Later: Ademollo et al, schoutens, Schwimmer, Seiberg

Def. (V. Kac - J. van de Leur, 1988). A superconformal algebra is a $\mathbb{Z}$-graded simple Lie superalgebra, $L = \sum_{i \in \mathbb{Z}} L_i$ such that

1) $\dim_{\mathbb{C}} L_i \leq d \;\; \forall i$
2) $Vir \subseteq L_{\bar{0}}$.

Examples.

1) $W(1:n) = \mathrm{Der}(\mathbb{C}[t,t^{-1}] \otimes G(n)) =$

$$= \{ f(t,\xi_1,\dots,\xi_n)\partial/\partial t + \sum_{i=1}^{n} f_i(t,\xi_1,\dots,\xi_n)\partial/\partial\xi_i \}$$

2) Let $\gamma \in \mathbb{C}$, $S(n,\gamma) = \{ d \in W(1:n) \mid \operatorname{div}(t^\gamma d) = 0 \}$ "$U(n)$-superconformal algebras"

3) Consider $\mathbb{C}[t,t^{-1}]$ with contact brackets

$[f,g] = f'g - fg'$ Neveu-Schwarz sector or

$[f,g] = t(f'g - fg')$ Ramond sector

The Grassmann algebra $G(n)$ is equipped with a Poisson bracket

$$[f,g] = (-1)^{|f|} \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_i},$$

$$G(n) = \sum_{i=0}^{n} G(n)_i, \quad [G(n)_i, G(n)_j] \subseteq G(n)_{i+j-2}$$

The contact bracket on $\mathbb{C}[t, t^{-1}]$ and the Poisson bracket on $G(n)$ extend to a contact bracket on

$$\Lambda(1:n) = \mathbb{C}[t, t^{-1}] \otimes G(n)$$

Let
$$K(1:n) = (\Lambda(1:n), [\,,\,])$$

simple, except for $n=4$, $[K(1:4), K(1:4)]$ is simple.

For $n=1 \Rightarrow$ Neveu-Schwarz and Ramond.

"$SO(n)$ - superconformal algebras"

4) $K^{(2)}(1:n) = \mathbb{C}[t^2, t^{-2}, \xi_1, \ldots, \xi_{n-1}] +$

$t\,\xi_n \mathbb{C}[t^2, t^{-2}, \xi_1, \ldots, \xi_{n-1}]$, Ramond bracket.

5) 1996, Cheng-Kac and Grozman-Leites-Shchepochkina : a new exceptional superalgebra $CK(6)$.

Conjecture (Kac - van de Leur) 1)-5) are all superconformal algebras.

## Lie Algebras vs Jordan Algebras.

$L$ Lie algebra, $L \supset sl_2(F) = Fe + Ff + Fh$, $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$.

Suppose that $L = L_{-2} + L_0 + L_2$ the sum of eigenspaces with respect to $ad(h)$.

For $x, y \in L_2$

$$x \circ y = [[x,f],y] \in L_2$$

J. Tits: $J = (L_2, \circ)$ is a Jordan algebra.

J. Tits, I. Kantor, M. Koecher: every Jordan algebra can be obtained in this way

$$J \longrightarrow TKK(J).$$

# Kantor Double.

Let $A$ be an associative commutative (super) algebra with a Poisson bracket $[a,b]$.

$$K(A) = A + Av,\ |v| = 1,\ [av, bv] = (-1)^{|b|}[a,b].$$

I. Kantor: $K(A)$ is a Jordan superalgebra.

There are brackets, that are not Poisson, but $K(A)$ is still a Jordan superalgebra.

For example: $[f,g] = f'g - fg'$

We call such brackets Jordan brackets.

Cantarini, Kac:

Jordan brackets $\xleftrightarrow{1-1}$ Contact brackets.

$$K(1:n) = TKK\Big(\overset{Kan(\mathbb{C}[t,t^{-1},\xi_1,\dots,\xi_{n-3}])}{\cancel{\mathbb{C}[t,t^{-1},\xi_1,\dots,\xi_{n-3}]}}\Big),\ n \ge 3$$

$$CK(6) = TKK(JCK(6)).$$

V. Kac, C. Martinez, E. Z., 2001: Jordan superconformal algebras.

In fact, Jordan superalgebras $J = \sum_{i \in \mathbb{Z}} J_i$, graded simple, $\dim_{\mathbb{C}} J_i \leq d$, $i \in \mathbb{Z}$.

Superconformal algebras over arbitrary associative commutative superalgebras (joint work with C. Martinez and Z. Zhang).

$F[t, t^{-1}] \rightsquigarrow A$.

1. $W(A:n) = \mathrm{Der}(A \otimes G(n))$

2. $\Delta : W(A:n) \to A$ is a divergence if

(i) $\Delta$ is an even 1-cocycle, i.e. a derivation into the module $A$;

(ii) $\Delta(a \cdot d) = a\Delta(d) + (-1)^{|a|\cdot|d|} d(a)$

Example. $d = \sum_{i=1}^{m} f_i \partial/\partial x_i + \sum_{j=1}^{n} h_j \partial/\partial \xi_j \in Der(F[x_1, \ldots, x_m] \otimes G(n))$

$$div(d) = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} + \sum_{j=1}^{n} (-1)^{|h_j|} \partial/\partial \xi_j (h_j).$$

Example. $A = \mathbb{C}[t, t^{-1}]$, $\gamma \in \mathbb{C}$.

$$\Delta(t^{i+1} \partial/\partial t) = (\gamma + i + 1) t^i, \quad i \in \mathbb{Z}$$

Let $\Delta : Der(A) \to A$ be a divergence.

Extend to $\Delta^{(n)} : Der(A \otimes G(n)) \to A \otimes G(n)$.

$$\tilde{S}(\Delta, n) = \{ d \in W(A : n) \mid \Delta^{(n)}(d) = 0 \}$$

$$S(\Delta, n) = [\tilde{S}(\Delta, n), \tilde{S}(\Delta, n)]$$

Example 2 $\Rightarrow$ classical superconformal algebras.

3. $A$, a contact superalgebra. Extend the contact bracket to $A \otimes G(n)$.

$$K(A:n) = [A \otimes G(n), A \otimes G(n)].$$

4. In 2001 C. Martinez - E. Z. : $CK(A, d)$,

$d : A \to A$ even derivation.

For $A = \mathbb{C}[t, t^{-1}]$, $d = \partial/\partial t$ or $t\,\partial/\partial t \Rightarrow CK(6)$.

5. $A = A^{(0)} + A^{(1)}$ $\mathbb{Z}/2\mathbb{Z}$-grading, automorphism of order 2

$K^{(2)}(A:n)$ = fixed elements in $K(A:n)$.

**Theorem** (C. Martinez - E. Z. - Z. Zhang)

Let $A$ be a finitely generated associative commutative superalgebra. Then the superalgebras 1.-5. have finitely presented universal central extensions.

Remark. There are some additional assumptions for $n = 1$ or $2$.

Corollary 1 (Theorem above + Kac - van de Leur).

All known superconformal algebras are finitely presented.

Corollary 2. $H(2m:n)$ is finitely presented.

## Representation Theory.

Cuspidal Modules: $V = \sum_{i \in \mathbb{Z}} V_i$, $\dim_F V_i < \infty$,

irreducible and both sets

$\{i > 1 \mid V_i \neq (0)\}$, $\{i < -1 \mid V_i \neq (0)\}$

are infinite.

O. Mathieu (1991). classification of cuspidal modules over Vir.

$V = \overline{F[t, t^{-1}]} = \sum_{i \in \mathbb{Z}} F \overline{t^i}$

$\not{c}$ acts as $0$ ;

$\exists\, \alpha, \beta \in F$ :

$$e_i\, \overline{t^j} = (j + \alpha i + \beta)\, \overline{t^{i+j}}$$

Theorem (C. Martinez, O. Mathieu, E. Z.)

Classification of all cuspidal modules over all known superconformal algebras.

arXive : 2505.20974, 172 p.

V. Kac - J. van de Leur : central extensions.

Nontrivial only for $W(1:1)$, $K(1:n)$, $n = 1, 2, 3, 4$.

In Mathieu's Theorem the central element $\mathfrak{c}$ of the Virasoro algebra acted trivially on cuspidal modules.

A new phenomenon : one 2-cocycle of $K(1:4)$ acts nontrivially.

$$K(1:3) \subseteq K(1:4) \leftarrow \widehat{K(1:4)} \subseteq CK(6)$$

$$\begin{tikzcd}[column sep=small] & \text{New 1-parametric families of representations} \arrow[dl] \arrow[dr] & \\ K(1:3) & & CK(6) \end{tikzcd}$$

$\widehat{K(1:4)}$ = Tits - Kantor - Koecher construction of the Jordan superalgebra $Kan(\mathbb{C}[t, t^{-1}, \xi])$ = Kantor double of $\mathbb{C}[t, t^{-1}, \xi]$.

This Jordan superalgebra

$Kan(\mathbb{C}[t, t^{-1}, \xi]) \hookrightarrow 2 \times 2$ matrices over the Weyl algebra

$$W = \langle t, t^{-1}, d \mid td - dt = 1 \rangle,$$

$\hat{K}(1:4) \hookrightarrow 4 \times 4$ matrices over $W$.

The $4 \times 4$ identity matrix = the central element that may act nontrivially.

Dear Manuel,

Happy Birthday!!!

Remark. There is life after 70.

I know it ...