Spectral Galois Theory for the Korteweg-de Vries variational Equation

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Categorical, homological and computational methods in Algebra
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I will present recent and ongoing joint work with

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Spectral problem

Schrödinger equation

$$-\Psi_{zz} + u(z)\Psi = E\Psi, \quad \text{or } (-\partial^2 + u)\Psi = E\Psi, \tag{1}$$

with $u(z)$ satisfying a stationary Korteweg de Vries (KdV) equation of the celebrated KdV hierarchy.
Now, we are going to focus in sKdV_1, the classical nonlinear **stationary KdV equation**:

$$u_{zzz} - 6uu_z = 0.$$

[Drach's Ideology, 1919:
Integrate (1) as an ODE to obtain a parametric solution $\Psi(x; E)$]

An example.

Take $u = 2\wp(z; g_2, g_3) = 2\wp(z)$ associated to the elliptic curve $\mathcal{E}$ defined by the polynomial $Y^2 = 4X^3 - g_2 X - g_3$

$$L = -\partial^2 + 2\wp(z) \quad \text{and} \quad A_3 = -4\partial^3 + 6u\partial + 3u' \,, \tag{2}$$

To solve the coupled spectral problem:

$$L\psi = E\psi \quad , \quad A_3\psi = \mu\psi \,. \tag{3}$$

Some extra facts:

$$[L, A_3] = LA_3 - A_3 L = 0 \quad , \quad A_3^2 + 16L^3 - 4g_2 L + 4g_3 = 0.$$

Then (L, A_3) behaves like a point on the algebraic curve defined by $\mu^2 + 16E^3 - 4g_2 E + 4g_3 = 0$, the *spectral curve* associated to the coupled spectral problem (3).

Algebro-geometric ODOs $\Leftrightarrow$ nontrivial centralizers

The theory of commuting ODOs has broad connections with many branches of modern mathematics:

- Non-linear partial differential equations (find new exact solutions).
- Algebra (the Dixmier or Jacobian or Poisson conjectures, highly non-trivial and still open).
- Complex analysis. Deformation quantisation. ...

[Zheglov, A., 2020. **Algebra, Geometry and Analysis of Commuting Ordinary Differential Operators**]

Our contributions to the commuting ODOs theory are framed in three items:

- Use effective differential algebra to develop symbolic algorithms: for parametric factorization of linear ordinary operators associated to spectral problems.
- Compute new pairs of commuting ODOs.
- Develop spectral Picard-Vessiot (PV) theory, i.e. PV theory for spectral problems.

For the Schrödinger operator (1), they are contained in a joint work by J.J Morales, S. L. Rueda, M. A. Zurro

- *Factorization of KdV Schrödinger operators using differential subresultants.* Adv. Appl. Math., 2020.
- *Spectral Picard-Vessiot fields for algebro-geometric Schrödinger operators* . Ann. Inst. Fourier, 2021.

Goals

Use an algebrogeometric approach to study the solutions of the variational equatio (VE)s around KdV solitons.

1. Compute the spectral problems associated with the method of separation of variables, and the spectral curve Γ.
2. Find the "relations" that allows transport solutions **from** the Lax pairs of Schrödinger operators **to** the Lax pair associated to VE.
3. Analyze the distinct type of solutions of the VE depending on the spectral curve Γ.

Next we show how to carry out this process for the cnoidal wave.

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The Lax pairs

The KdV equation we studied is

$$u_t = 6uu_x - u_{xxx}, \text{ with } u = u(x,t).$$

1. It can be obtained as the compatibility condition of the Lax pair of differential operators

$$L := -\partial_{xx} + u \quad , \quad A := -4\partial_{xxx} + 6u\partial_x + 3u_x,$$

with Lax equation $L_t = [A, L]$, i. e. and $u_t = [A, L]$.

2. In this situation we can consider the differential system

$$L\phi = E\phi\,,\ \phi_t = A\phi\,, \tag{4}$$

3. If we try to find solutions of problem (4) in separate variables of the form $\phi(x,t) = e^{\mu t}\varphi(x)$, then φ must be a solution of the following spectral problem.

$$L\varphi = E\varphi \quad , \quad A\varphi = \mu\varphi.$$

Traveling waves solutions: $u = u(z)$ with $z = x - ct$, $t = t$.

$$u_t = 6uu_z - u_{zzz} + cu_z, \tag{5}$$

Then, the Lax pair is

$$L\phi = E\phi\,,\ \phi_t = \tilde{A}\phi\,, \tag{6}$$

with $\partial = \partial_z$ and $\tilde{A} = -4\partial^3 + (6u + c)\partial + 3u_z$.

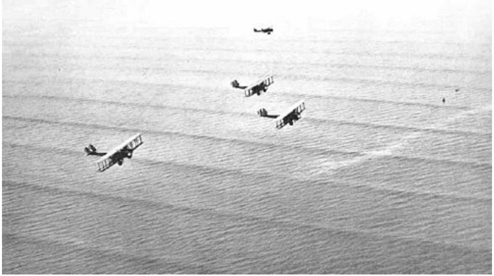
Goal: Find solutions of problem (6) in separate variables of the form $\phi(z,t) = e^{\mu t}\varphi(z)$, then φ must be a solution of the following spectral problem.

$$L\varphi = E\varphi \quad , \quad \tilde{A}\varphi = \mu\varphi. \tag{7}$$

The cnoidal wave

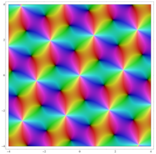
From Wikipedia

"US Army bombers flying over near-periodic swell in shallow water, close to the Panama coast (1933). The sharp crests and very flat troughs are characteristic for cnoidal waves"



The cnoidal wave

- In fluid dynamics, a cnoidal wave is a periodic, nonlinear, and exact wave solution of the Korteweg-de Vries equation. These solutions are expressed in terms of the Jacobi elliptic function cn, hence they are called *cnoidal waves*.
- The cnoidal wave can be defined by the Weierstrass function $\wp$



Visualization of the $\wp$-function with invariants $g_2 = 1 + \imath$, $g_3 = 2 - 3\imath$, in which white corresponds to a pole, black to a zero.

The cnoidal wave

Recall that the cnoidal wave can be defined by the potential $u_0(z) = 2\wp(z) - \frac{c}{6}$ where $\wp(z)$ is the Weierstras function $\wp(z; g_2, g_3)$ associated to the elliptic curve $\mathcal{E}$ defined by the polynomial $Y^2 = 4X^3 - g_2 X - g_3$.

We denote by $\omega_1, \omega_2, \omega_3 := \omega_1 + \omega_2$ its half-periods. Then

$$e_1 = \wp(\omega_1; g_2, g_3)\,,\ e_2 = \wp(\omega_3; g_2, g_3)\,,\ e_3 = \wp(\omega_2; g_2, g_3)\,,$$

are the roots of the polynomial $4X^3 - g_2 X - g_3$, and the half-periods are the roots of $\wp'$ in a fundamental period paralelogram Ω,

$$\wp'(\omega_i; g_2, g_3) = 0\ ,\ \ i = 1, 2, 3, \tag{8}$$

[G. Pastras, Four Lectures on Weierstrass Elliptic Function and Applications in Classical and Quantum Mechanics, Arxiv, 2017.]

The Weierstrass functions

Fixed a two-dimensional lattice $\Lambda = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z} \subset \mathbb{C}$, $\Lambda^* = \Lambda \setminus \{0\}$

- Weierstrass elliptic function:
  $$\wp(z;\Lambda) := \wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*}\left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right).$$
  AND $\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3$.
- The Weierstrass zeta function:
  $$\zeta(z;\Lambda) = \frac{1}{z} + \sum_{w\in\Lambda^*}\left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right).$$
  AND $\zeta' = -\wp$.
- Weierstrass sigma function:
  $$\sigma(z;\Lambda) = z\prod_{w\in\Lambda^*}\left(1-\frac{z}{w}\right)\exp\left(\frac{z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2\right)$$
  AND $\frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$, then $\frac{d^2}{dz^2}\log\sigma(z) = -\wp(z)$.

The spectral curve for the cnoidal wave

We can compute the algebraic curve associated to the spectral problem (7): We can normalize with the change of parameter $\tilde{E} := E + \frac{c}{6}$ and then

- $L - E = -\partial^2 + 2\wp(z) - \tilde{E}$
  the Lamé equation: $-\varphi'' + 2\wp(z)\varphi = \tilde{E}\varphi$.
- The associated spectral curve Γ for the pair of operators $L, \tilde{A}$ is defined by $\det(W - \mu) = \mu^2 - P(\tilde{E})$ $\mathbb{C}^2$, where

$$\begin{aligned} \psi_z &= \begin{pmatrix} 0 & 1 \\ 2\wp(z) - \tilde{E} & 0 \end{pmatrix} \psi = U\psi, \\ \mu\psi &= \begin{pmatrix} -2\wp'(z) & 4\wp(z) + 4\tilde{E} \\ 4\wp(z) - 4\wp^2(z) - 4\tilde{E}^2 - g_2 & 2\wp'(z) \end{pmatrix} \psi = W\psi. \end{aligned}$$

Solutions of the Lamé equation

Specifically,

$$\Gamma : \ \mu^2 = P(\tilde{E}) \quad , \quad \text{with} \quad P(\tilde{E}) := -16\tilde{E}^3 + 4g_2\tilde{E} - 4g_3. \quad (9)$$

The parametrization of the spectral curve: $\aleph(s) = (\tilde{E}(s), \mu(s))$

$$\tilde{E}(s) := \wp(\imath \cdot s)\,,\ \mu(s) := -2\imath \cdot \wp'(\imath \cdot s)\ ,$$

A fundamental matrix at each point of Γ:

- At a non branch point of Γ

$$\begin{aligned}\boldsymbol{\Phi}(x,t,s) &= \boldsymbol{\Psi}(x-ct,s)\begin{pmatrix} e^{\mu(s)t} & 0 \\ 0 & e^{-\mu(s)t}\end{pmatrix} \\ &= \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_1' & \rho_2' \end{pmatrix}\begin{pmatrix} e^{\left(2\wp'(s)+c\zeta(s)\right)t-\zeta(s)x} & 0 \\ 0 & e^{-\left(2\wp'(s)+c\zeta(s)\right)t+\zeta(s)x}\end{pmatrix}\end{aligned}$$

with $\rho_i = \dfrac{\boldsymbol{\sigma}(x-ct+(-1)^{i+1}s)}{\boldsymbol{\sigma}(s)\boldsymbol{\sigma}(x-ct)}$ and $\boldsymbol{\sigma}$ and ζ are the Weiertrass sigma function and the Weiertrass zeta function respectively.

Solutions of the Lamé equation

Next, we have to solve at points in $Z := \Gamma \cap \{\mu = 0\}$. Hence, we consider the operator $-\partial_z^2 + 2\wp(z) + e_i$ where $P(-e_i) = 0$

- At a branch point $P = (-e_i, 0) \in Z$ and from the *Lamé solutions*, we obtain

$$\mathbf{\Phi}(x,t) = \mathbf{\Psi}(x-ct)\begin{pmatrix} e^{0\cdot t} & 0 \\ 0 & e^{-0\cdot t} \end{pmatrix} = \begin{pmatrix} \widehat{\varphi_{i,e_i}} & \varphi_i \\ (\widehat{\varphi_{i,e_i}})' & \varphi_i' \end{pmatrix}.$$

with $\widehat{\varphi_{i,e_i}} := 1/\sqrt{\wp(z) - e_i}$ and $\varphi_i = \sqrt{\wp(z) - e_i}$, and $z = x - ct \in \Omega \setminus \{\omega_i\}$.

Contents

KdV Variational equation

Consider $u_0 + \epsilon\xi$ and substitute in the equation (5)

$$u_\tau = -u_{zzz} + 6uu_z + cu_z$$

Equating in ϵ we obtain the KdV variational equation (VE)

$$\xi_\tau = 6u_{0,z}\xi + (6u_0 + c)\xi_z - \xi_{zzz}$$

The variational equation

For $u_0(z) = 2\wp(z) - \frac{c}{6}$, the *variational equation* considered is the differential equation:

$$\xi_\tau = M\xi, \tag{10}$$

where $M = \partial_z L_0$ and $L_0 = -\partial_z^2 + (6u_0 + c) = -\partial_z^2 + 12\wp(z)$.
We confront the study of the third-order operator

$$M = -\partial_z^3 + 12\wp(z)\partial_z + 12\wp'(z). \tag{11}$$

The (formal) *adjoint variational equation* is now

$$\psi_\tau = M^\dagger\psi, \quad \text{with } M^\dagger = L_0\partial_z = -\partial_z^3 + 12\wp(z)\partial_z. \tag{12}$$

To solve the equation (12), we will use the base of solutions obtained previously associated to the spectral problems of the Schrödinger operator and its spectral Γ curve defined in (9).

Relation of the solutions spaces

$$\begin{array}{ccc} \psi \text{ solution of } (12) & \xrightarrow{\partial_z} & \xi := \partial_z\psi \text{ solution of } (11) \\ \psi := \partial_z^{-1}\xi \text{ solution of } (12) & \xleftarrow{\partial_z^{-1}} & \xi \text{ solution of } (11) \end{array}$$

In particular:

- If $\psi(z,\tau)$ is a solution of (12), then $\xi(z,\tau) := \partial_z\psi(z,\tau)$ is a solution of (11).
- We have an algebraic tools to solve (12):
  The second symmetric power of $L - E$.

The second symmetric power of $L - E$

- The *second symmetric power of* $L - E$, denoted by $(L-E)^{\odot 2}$, is defined as a nonzero element of the differential operator ring with coefficients in the differential ring of polynomials in E with coefficients rational functions in u_0 of minimal order such that $(L-E)^{\odot 2}(\varphi_i^2) = 0$ where $\{\varphi_1, \varphi_2\}$ is a basis of solutions of $L - E$.
- Computing:

$$(L-E)^{\odot 2} = -\partial_{zzz} + 4(u_0 - E)\partial_z + 2u_{0,z}$$

- Define $\psi := \phi^2$, thus $\psi_{zzz} = 4(u_0 - E)\psi_z + 2u_{0,z}\psi$.

- By (6), the time evolution of ϕ is given by $\phi_t = -4\phi_{zzz} + (6u_0 + c)\phi_z + 3u_{0,z}\phi$, and then we find that ψ is a solution of the following linear differential system of PDE

$$\begin{cases} -\psi_{zzz} + 4(u_0 - E)\psi_z + 2u_{0,z}\psi & = 0\,, \\ -\psi_t + (2u_0 + 4E + c)\psi_z - 2u_{0,z}\psi & = 0. \end{cases}$$

Then, eliminating E in this system, we obtain

$$\psi_t + \psi_{zzz} - (6u_0 + c)\psi_z = \psi_t + \psi_{zzz} - 12\wp(z)\psi_z = 0$$

Then we can recover the adjoint variational equation.

Closed form solutions of VE

We detail the formulas that we have obtained in the case of the cnoidal wave.

- At non branch points of Γ, we have obtained a basis of solution $\{\psi_1, \psi_2, \psi_3\}$ with

$$\psi_i = \phi_i(z,s)^2 = e^{2\mu(s)t}\rho_i^2(z)\ , \quad i = 1, 2,$$

$$\psi_3 = \phi_1\phi_2 = \frac{\boldsymbol{\sigma}(z+s)\boldsymbol{\sigma}(z-s)}{\boldsymbol{\sigma}(s)^2\boldsymbol{\sigma}(z)^2} = \wp(s) - \wp(z)\ ,$$

  for $(z,s) \in (\Omega \setminus \{\omega_1, \omega_2, \omega_3\}) \times (\Omega \setminus \{\omega_1, \omega_2, \omega_3\})$.

- At the branch points $B_j = (-e_j, 0)$, $j = 1, 2, 3$, we have obtained a basis of solution $\{\psi_{1,e_j}, \psi_{2,e_j}, \psi_{3,e_j}\}$ with

$$\psi_{1,e_j}(z) = (\widehat{\varphi_{j,e_j}})^2\ ,\ \psi_{2,e_j}(z) = (\varphi_{j,e_j})^2\ ,\ \psi_{3,e_j} = 1\ ,$$

  for $z \neq \omega_j$ in a fundamental domain Ω

Closed form solutions of VE

Consequently, we obtain the following formulas for the corresponding solutions of the variational equation $\xi_\tau = M\xi$.

1. At non branch points of Γ, we have obtained a basis of solution $\{\widetilde{\psi_1}, \widetilde{\psi_2}, \widetilde{\psi_3}\}$ with

$$\widetilde{\psi}_i = \partial_z(\psi_i), \quad i = 1, 2, \quad \widetilde{\psi_3} = \partial_z(\psi_3) = -\wp'(z) \ ,$$

   for $(z, s) \in (\Omega \setminus \{\omega_1, \omega_2, \omega_3\}) \times (\Omega \setminus \{\omega_1, \omega_2, \omega_3\})$.

2. At the branch points $B_j = (-e_j, 0)$, $j = 1, 2, 3$, we have obtained a basis of solution $\{\widetilde{\psi_{1,e_j}}, \widetilde{\psi_{2,e_j}}\}$ with

$$\widetilde{\psi_{1,e_j}} = \partial_z(\psi_{1,e_j}) = \frac{-\wp'(z)}{(\wp(z) - e_j)^2} \ ,$$

$$\widetilde{\psi_{2,e_j}} = \partial_z(\psi_{2,e_j}) = -\wp'(z) \ ,$$

   for $z \neq \omega_j$ in a fundamental domain Ω.

Solutions of VE ...

The solution of the variational equation along the cnoidal wave is studied in [Bottman, B. Deconinck, KdV Cnoidal waves are spectraly stable, Discrete and Continuous Dynamical Systems 25, 1163-1180 (2009)] also using the squared eigenfunction connection, but without any reference to the *Halphen-Hermite spectral bundle*.

But, in order to obtain, from the above, all the solutions of the variational equations on a space defined by suitable boundary conditions, it is necessary to prove a completeness statement.

Advantage:

- Effective computation of solution Υ.

$$K\langle\wp(s)\rangle = \mathbb{C}\langle u_0\rangle\langle\wp(s)\rangle \subset K\langle\wp(s)\rangle(\Upsilon).$$

- We describe the analytic character of the computed solution.
- PROBLEM: Closed form solutions of VE for general τ?

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Spectral curve of an operator L

Let Σ be an ordinary differential field, C the field of constants, assumed algebraically closed of characteristic zero.
Generalized Schur's Theorem [Goodearl, 1983]

$$\mathcal{Z}((M)) = \left\{ \sum_{j=-\infty}^{m} c_j Q^j \mid c_j \in C, m \in \mathbb{Z} \right\}, \text{ with } Q = M^{1/3}.$$

Commutative differential domain

$$\mathcal{Z}(M) = \mathcal{Z}((M)) \cap \Sigma[\partial]$$

Spec$(\mathcal{Z}(M))$ is an abstract algebraic curve Γ

Compute the defining ideal of Γ

Centralizer $\mathrm{ord}(\mathrm{M}) = 3$

- Given $M \in \Sigma[\partial] \setminus C[\partial]$, with $\mathcal{Z}(M) \neq C[M]$.
- $\mathcal{Z}(M)$ is a free $C[M]$-module of rank 3.
- $\{1, A_1, A_2\}$ basis of $\mathcal{Z}(M)$ as a $C[M]$-module. Each A_i is a monic operator in $\mathcal{Z}(M) \setminus C[M]$ of minimal order

$$o_i := \mathrm{ord}(\mathrm{A_i}) \equiv \mathrm{i} \pmod 3.$$

$$\mathcal{Z}(M) = C[M] \oplus C[M]A_1 \oplus C[M]A_2 = C[M, A_1, A_2]$$

BC ideal $\mathrm{ord}(M) = 3$

Consider:

$$e_M : C[\lambda, \mu_1, \mu_2] \to \Sigma[\partial]$$

$$e_M(\lambda) = M, \;\; e_M(\mu_1) = A_1, \;\; e_M(\mu_2) = A_2.$$

Image of e_M,

$$\mathcal{Z}(M) = C[M, A_1, A_2]$$

Given $g \in C[\lambda, \mu_1, \mu_2]$ denote

$$g(M, A_1, A_2) := e_M(g).$$

We define the Burchnall-Chaundy ideal of M:

$$\mathtt{BC}(M) := \mathrm{Ker}(\mathrm{e_M}) = \{\mathrm{g} \in \mathrm{C}[\lambda, \mu_1, \mu_2] \mid \mathrm{g}(\mathrm{M}, \mathrm{A_1}, \mathrm{A_2}) = 0\}.$$

Spectral curve for $\mathrm{ord(M)} = 3$

- $\mathrm{ord(M)} = 3$ in $\Sigma[\partial]$, $\mathcal{Z}(M) = C[M, A_1, A_2]$, $\mathrm{ord(A_i)} \equiv \mathrm{i} \bmod 3$
- Computing differential resultants:

$$\begin{cases} f_i = \partial\mathrm{Res}(M - \lambda, A_i - \mu_i), & i = 1, 2 \\ f_3^r = \partial\mathrm{Res}(A_1 - \mu_1, A_2 - \mu_2) \end{cases}$$

Moreover f_i are irreducible in $C[\lambda, \mu_1, \mu_2]$ since $\mathtt{BC}(L, A_i) = (f_i)$ and $\mathtt{BC}(A_1, A_2) = (f_3)$. Then we obtain:

$$(0) \subset (f_i) \subset (f_1, f_2) \subseteq (f_1, f_2, f_3) \subseteq \mathtt{BC}(L), \quad i = 1, 2.$$

- Define $\Gamma := V(\mathtt{BC}(M))$ and observe that

$$\Gamma \ \subseteq \ \gamma := V(f_1, f_2, f_3) \ \subseteq \ \beta := V(f_1, f_2).$$

Space algebraic curve β is the intersection of the irreducible surfaces defined by $f_1(\lambda, \mu_1) = 0$ and $f_2(\lambda, \mu_2) = 0$.

Spectral curve for $\mathrm{ord}(\mathrm{M}) = 3$

Theorem: (RZ 2023) $\mathtt{BC}(M)$ is a prime ideal

$$\mathtt{BC}(M) = (f_1, f_2, f_3)$$

Irreducible affine algebraic curve in C^3

$$\Gamma = V(\mathtt{BC}(M))$$

$$\mathcal{Z}(M) \simeq C[\Gamma] = \frac{C[\lambda, \mu_1, \mu_2]}{\mathtt{BC}(M)}$$

New coefficient field

$[\mathtt{BC}(M)]$ is a prime differential ideal of $\Sigma[\lambda, \mu_1, \mu_2]$

Differential domain

$$\Sigma[\Gamma] = \frac{\Sigma[\lambda, \mu_1, \mu_2]}{[\mathtt{BC}(M)]}$$

Its fraction field

$$\Sigma(\Gamma)$$

is a differential field with the extended derivation.

The intrinsic right factor

$\mathrm{ord}(\mathrm{M}) = 3$ in $\Sigma[\partial]$, $\mathcal{Z}(M) = C[L, A_1, A_2]$

Theorem: (RZ 2023) The greatest common right divisor in $\Sigma(\Gamma)[\partial]$

$$\partial + \phi = \mathrm{gcrd}(\mathrm{M} - \lambda, \mathrm{A}_1 - \mu_1, \mathrm{A}_2 - \mu_2)$$

equals $\mathrm{gcrd}(\mathrm{M} - \lambda, \mathrm{A}_1 - \mu_1) = \mathrm{gcrd}(\mathrm{M} - \lambda, \mathrm{A}_2 - \mu_2)$ and divides $\mathrm{gcrd}(\mathrm{A}_1 - \mu_1, \mathrm{A}_2 - \mu_2))$.

Assume $M = \partial^3 + u_1\partial + u_0$, then we obtain

$$M - \lambda = \left(\partial^2 - \phi\partial + u_1 - 2\phi' + \phi^2\right) \cdot (\partial + \phi),$$

in $\Sigma(\Gamma)[\partial]$, under the condition

$$\phi^3 + u_1\phi - 3\,\phi\phi' - u_0 + \phi'' + \lambda = 0.$$

Planar / Non planar spectral curves

1. (Dickson, Gesztesy, Unterkofler, 1999), $\Sigma = \mathbb{C}(x), \partial = d/dx$,
$$L = \partial^3 - \frac{15}{x^2}\partial + \frac{15}{x^3} + h \ .$$
$$\mathcal{Z}(L) = C[L, A_1, A_2] \ , \ \mathrm{ord(A_1) = 4, ord(A_2) = 8},$$
but we have $A_2 = A_1^2$.
2. (RZ 2022), $\Sigma = \mathbb{C}(x), \partial = d/dx$
$$L = \partial^3 - \frac{6}{x^2}\partial + \frac{12}{x^3} + h \ , \ \ h \in \mathbb{C}.$$
$\mathcal{Z}(L) = \mathbb{C}[L, A_1, A_2]$ with $\mathrm{ord(A_1) = 4,\ ord(A_2) = 5}$.

First explicit example of a non-planar spectral curve.

The curve defined by $\mathtt{BC}(L)$ is a non-planar curve Γ parametrized by
$$\aleph(\tau) = (h - s^3, s^4, -s^5), \quad s \in \mathbb{C}.$$

Non-planar spectral curve

Consider $h = 1$:

$$M = \partial^3 - \frac{6}{x^2}\partial + \frac{12}{x^3} + 1\ .$$

The spectral curve:

$$\aleph(s) = (1 - s^3, s^4, -s^5),\ s \in \mathbb{C}.$$

The first differential subresultants of $M - \lambda$, $A_1 - \mu_1$ and $A_2 - \mu_2$ pairwise are equal to

$$\phi_{i,0} + \phi_{i,1}\partial,\ \ i = 1, 2, 3,\ \ j = 0, 1,$$

with

$$\begin{aligned}
\phi_{1,0} &= (1-\lambda)\mu_1 - \tfrac{4\mu_1}{x^3} + \tfrac{8(\lambda-1)}{x^4}, & \phi_{1,1} &= (\lambda-1)^2 - \tfrac{2\mu_1}{x^2} + 4\tfrac{(\lambda-1)}{x^3},\\
\phi_{2,0} &= (1-\lambda)^3 - \tfrac{4(1-\lambda)^2}{x^2} + \tfrac{8\mu_2}{x^4}, & \phi_{2,1} &= (\lambda-1)^3 - \tfrac{4(1-\lambda)^2}{x^2} + \tfrac{8\mu_2}{x^3},\\
\phi_{3,0} &= -\mu_2\left(\mu_1^2 + \tfrac{4\mu_2}{x^3} - \tfrac{8\mu_1}{x^4}\right), & \phi_{3,1} &= \mu_1^3 - \tfrac{2\mu_2^2}{x^2} + \tfrac{4\mu_2\mu_1}{x^3}.
\end{aligned}$$

We have $\mathrm{ord}(\mathrm{A}_1) = 4$ and $\mathrm{ord}(\mathrm{A}_2) = 5$ thus

$$\phi = \overline{\phi}_i = \frac{\phi_{0,i}}{\phi_{1,i}} + [\mathtt{BC}(M)], \quad i = 1, 2, 3.$$

$$\phi(s) := \phi_i(\aleph(s)) = \frac{-s^3x^3 + 2s^2x^2 - 4sx + 4}{(s^2x^2 - 2sx + 2)x}.$$

Thus, $M + s^3 - 1 = \partial^3 - \frac{6}{x^2}\partial + \frac{12}{x^3} + s^3$, and

$$M + s^3 - 1 = \left(\partial^2 + \phi(s)\partial + \phi(s)^2 + 2\phi(s)' - \frac{6}{x^2}\right) \cdot (\partial + \phi(s)).$$

Thus we can compute a solution of $\partial + \phi(s) = 0$ to obtain

$$\varphi(x) = -sx - \ln\left(s^2x^2 - 2\,sx + 2\right) + 2\,\ln(x) + C_1$$

Contents

Around the cnoidal wave

- $\xi_\tau = M\xi$, $\xi = e^{\nu\tau}\tilde{\psi}(z)$
- $M = -\partial^3 + 12\wp(z)\partial + 12\wp'(z)$, and the spectral problem

$$M\tilde{\psi} = \nu\tilde{\psi}$$

.

The spectral curve that support the line bundle:

$$\tilde{\Gamma} := \mathrm{Spec}(C[M, A_1, A_2])$$

with defining ideal $\mathtt{BC}(M) = (f_1, f_2, f_3)$, which can be calculated using the work done with Sonia Rueda on the third order operator. Pretty long formulas!!!

Computing

- The operators for the cnoidal wave:

$$A_1 = \partial^7 - 28\wp(z)\partial^5 - 84\wp'(z)\partial^4 - 560\wp(z)\partial^3 - 560\wp(z)\wp'(z)\partial^2 + \left(-6720(\wp(z))^3 + 1400(\wp'(z))^2 + 2352\wp(z)g_2 + 1560g_3\right)\partial + 784\wp'(z)g_2$$

$$A_2 = \partial^5 - 20\wp(z)\partial^3 - 40\wp'(z)\partial^2 + \left(-120(\wp(z))^2 + 18g_2\right)\partial + 1.$$

The right factor $\partial + \phi$ whose solution is $\tilde{\psi}(z)$ is given by a formula depending on (z, P) for $P \in \tilde{\Gamma}$. Then

$$\xi = e^{\nu\tau}\tilde{\psi}(z, \nu, \mu_1, \mu_2)$$

with $P = (\nu, \mu_1, \mu_2) \in \tilde{\Gamma}$ and $\nu \neq 0$.

Contents

Upcoming research

- J. J. Morales-Ruiz, J. P. Ramis, Z.. *Variational equation around KdV Traveling waves and Differential Galois Theory*, In preparation.
- R. Delgado, R. Hernandez Heredero, Z.. *Rational pseudodifferential operators and integrability conditions*, Project in progress.
- I am working on the full *intrinsic factorization* of the third order operator depending on the differential field of its coefficients.
    - Parametric Picard-Vessiot Theory (PPV) is needed for this.
    - A new differential tool is needed: *PPV over spectral curves !*

Congratulations Manuel!!!